**Exercise 1.*** Let $(R, \mathfrak{m}, k)$ be a local ring and $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ be a regular element. If $M$ is an $R/(x)$-module, prove that

$$\beta_i^R(M) = \beta_i^{R/(x)}(M) + \beta_{i-1}^{R/(x)}(M) \quad \text{for each} \quad i \geqslant 0\,.$$

**Hint:** Let $F$ be the minimal resolution of $M$ over $R$. A lemma from lecture says that $F$ is the minimal semifree resolution of $M$ over $E = R[e \mid \partial e = x]$, now consider $F \otimes_E R/(x)$.

**Exercise 2.*** Let $R = Q/I$ where $Q$ is a local ring and $I$ is minimally generated by $f_1, \ldots, f_n$. If $f_1$ is a $Q$-regular element, prove that the map $I/I^2 \to R$ given by

$$a_1 f_1 + \cdots + a_n f_n \mapsto a_1$$

is a well-defined $R$-linear map and determines a free $R$-summand of $I/I^2$.

**Exercise 3.*** Let $R$ be a noetherian local ring with minimal regular presentation $\widehat{R} \cong Q/I$ and $Q[X]$ a minimal model for $R$ over $Q$. Let $\theta\colon Q[X] \longrightarrow Q[X]$ be a derivation of degree $d \leqslant 0$ such that $\theta((X_{\geqslant 0})) \subseteq (X_{\geqslant 0})$. Show that $\theta$ induces a well-defined derivation $\theta_n\colon k[X_{\geqslant n}] \longrightarrow k[X_{\geqslant n}]$.

**Exercise 4.** Let $R$ be a noetherian local ring with minimal regular presentation $\widehat{R} \cong Q/I$ and $Q[X]$ a minimal model for $R$ over $Q$. Consider the map $\theta_z = \left[\partial, \frac{d}{dz}\right]$.

a) Show that $\theta_z\colon Q[X] \longrightarrow Q[X]$ is a chain derivation.

b) Show that $\operatorname{im}(\theta_z) \subseteq (X)$.

**Exercise 5.*** Let $R = Q/I$ with $Q = k[\![x, y, z]\!]$ and $I = (x^2, xy)$. Compute $R \otimes_{Q[X]} \Omega_{Q[X]|Q}$ up to homological degree 3.

**Exercise 6.** Let $R = \mathcal{C}^\infty(\mathbb{R}^n)$ be the ring of smooth functions on $\mathbb{R}^n$, and $\mathfrak{m}$ be the maximal ideal of functions that vanish at a fixed point $x_0 \in \mathbb{R}^n$. Show that

$$\operatorname{Der}_{\mathbb{R}}(R, R/\mathfrak{m}) \cong \operatorname{Hom}_{\mathbb{R}}(\mathfrak{m}/\mathfrak{m}^2, R/\mathfrak{m}) \cong \mathbb{R}^n$$

as vector spaces. As a moral, we conclude that $\operatorname{Der}_{\mathbb{R}}(R, R/\mathfrak{m})$ is a model of the tangent space of $\mathbb{R}^n$ at $x_0$ constructed from the ring of smooth functions.