Exercise 1.* Let R be a noetherian local ring with minimal regular presentation $\widehat{R} \cong Q/I$ and $Q[X]$ a minimal model for R over Q. Let $\theta\colon Q[X] \longrightarrow Q[X]$ be a derivation of degree $d \leqslant 0$ such that $\theta((X_{\geqslant 0})) \subseteq (X_{\geqslant 0})$. Show that θ induces a well-defined derivation $\theta^{(n)}\colon k[X_{\geqslant n}] \longrightarrow k[X_{\geqslant n}]$. Moreover, show that if θ is a chain derivation then $\theta^{(n)}$ is a chain derivation.

Exercise 2. Let $R = Q/I$ with $Q = [\![x,y,z,w]\!]$ and $I = (xy, z^2, zw, x^2z + w^3, y^2w + xw^2)$. Compute $R \otimes_{Q[X]} \Omega_{Q[X]|Q}$ up to homological degree 3.

Exercise 3.* Let R be a noetherian local ring with minimal regular presentation $\widehat{R} \cong Q/I$ and $Q[X]$ a minimal model for R over Q. Fix $z \in X$. Recall that $\frac{d}{dz}\colon Q[X] \longrightarrow Q[X]$ is the derivation defined on variables $x \in X$ by

$$\frac{d}{dz}(x) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise} \end{cases}$$

Let $u \in X$ with $u \neq z$.

a) Show that
$$\frac{d}{dz}(zu) = u \qquad \text{and} \qquad \frac{d}{dz}(uz) = (-1)^{|z||u|}u.$$

b) Show that if $|z|$ is even, then
$$\frac{d}{dz}(z^2) = 2z.$$

Exercise 4. Let R be a noetherian local ring with minimal regular presentation $\widehat{R} \cong Q/I$ and $Q[X]$ a minimal model for R over Q. Consider the map $\theta_z = \left[\partial, \frac{d}{dz}\right]$.

a) Show that $\theta_z\colon Q[X] \longrightarrow Q[X]$ is a chain derivation.

b) Show that $\operatorname{im}(\theta_z) \subseteq (X) = (X_{\geq 0})$.

Exercise 5.* Let R be a noetherian local ring with minimal regular presentation $\widehat{R} \cong Q/I$ and $Q[X]$ a minimal model for R over Q. Given an element $z \in X$, describe $\theta_z(x)$ for $x \in X$.

Exercise 6. Let R be a noetherian local ring with minimal regular presentation $\widehat{R} \cong Q/I$ and $Q[X]$ a minimal model for R over Q. Show that for all $y, z \in X$, if $y < z$ then

$$\widetilde{\theta}_z(y^\vee) = -[z^\vee, y^\vee].$$

Note that we have already shown this holds when $z < y$ in lecture.

Exercise 7.* Let $R = Q/I$ be a minimal regular presentation and $A = Q[X] \xrightarrow{\simeq} R$ the minimal model. Let $R[X] = R \otimes_Q A$, note this is a semifree dg R-algebra.

a) Prove that $\mathrm{H}_*(R[X]) = \operatorname{Tor}_*^Q(R,R)$ as graded R-modules.

b) Show that $(X) := (X_{\geqslant 1}) \subseteq R[X]$ is a dg ideal.

c) Show that $(X)/(X)^2 \cong R \otimes_A \Omega_{A/Q}$ as complexes of R-modules.

d) Prove that $\mathrm{H}_1((X)/(X)^2) = I/I^2$.

Hint: Note that $Q[X]_{\geqslant 1} \to I$ is a dg Q-algebra resolution. Consider $R[X]_{\geqslant 1} = Q[X]_{\geqslant 1} \otimes_Q R$.

Exercise 8. Let $R = Q/I$ be a regular presentation and $Q[X]$ a minimal model for R over Q. There exists a unique derivation $\varepsilon\colon Q[X] \to Q[X]$ such that $\varepsilon(x) = x$ for all $x \in X_{\geq 1}$. What is $\varepsilon(x_{i_1} \cdots x_{i_n})$ where $x_{i_1}, \ldots, x_{i_n} \in X_{\geq 1}$?

Exercise 9. Find a counterexample that shows that Exercise 2 on yesterday's problem set (Problem Set 6) was false as stated (oops!). The corrected statement should have asked for f_1 to be regular on $Q/(f_2, \ldots, f_n)$, to force the desired map to determine a R-free summand of I/I^2.