Exercise 1.* Let $(R, \mathfrak{m}, k)$ be a local ring. We know that there exists a graded Lie algebra L with $\operatorname{rank}_k(L_n) = \varepsilon_n(R)$ and $\operatorname{Ext}_R(k,k) \cong \mathrm{U}L$ as graded k-algebras. Complete the proof of the Poincaré-Birkhoff-Witt theorem for $L = \pi^*(R)$ by showing that the set of admissible monomials is linearly independent.[1]

Exercise 2.* Let $R = k[x]/(x^2)$ and prove that $\operatorname{Kos}^R(x)$ has the same homology as $k \oplus \Sigma k$, but these complexes are **not** isomorphic in $\mathsf{D}(R)$.

Exercise 3.* Let A be a graded commutative dg Q-algebra and $z \in A$ cycle. Introduce a variable t where $|t| = |z| + 1$, and consider A as a dg $Q[t]$-algebra with $t \mapsto z$ and $Q[t]$ acts on Q by $tQ = 0$. Prove that

$$A \otimes^{\mathsf{L}}_{Q[t]} Q \cong A\langle x \mid \partial x = z \rangle\,.$$

Exercise 4. Let A be a dg algebra and consider a triangle $X \to Y \to Z \to \Sigma X$. From lecture this corresponds to a map of semifree dg A-modules $f\colon F_X \to F_Y$ where $\operatorname{cone}(f)$ is a semifree resolution of F_Z.

(a) For any dg A-module M, prove that there is a long exact sequence

$$\cdots \to \operatorname{Hom}_{\mathsf{D}(A)}(M, \Sigma^{i-1} Z) \to \operatorname{Hom}_{\mathsf{D}(A)}(M, \Sigma^i X) \to \operatorname{Hom}_{\mathsf{D}(A)}(M, \Sigma^i Y) \to \operatorname{Hom}_{\mathsf{D}(A)}(M, \Sigma^i Z) \to \cdots\,.$$

(b) Use part (a), to show there is a long exact sequence

$$\cdots \to \mathrm{H}_{i+1}(Z) \to \mathrm{H}_i(X) \to \mathrm{H}_i(Y) \to \mathrm{H}_i(Z) \to \mathrm{H}_{i-1}(X) \to \cdots\,.$$

Exercise 5. Let $Q = k[\![x,y]\!]$ and $I = (x^2, xy)$. Find a system of higher homotopies for multiplication by xy on the minimal free resolution F for Q/I over Q.

Exercise 6.* Recall that a *matrix factorization* for a polynomial f is a pair of matrices A, B such that AB and BA are both equal to $f \cdot \mathrm{Id}$ (for Id the appropriate identity matrix).

(a) Prove that if $f \in Q \smallsetminus \{0\}$ with Q a regular local ring, then A and B are square matrices.

(b) Find a matrix factorization for $f = x^3 + y^4$, corresponding to the tail of the free resolution of k over $R = k[\![x,y]\!]/(f)$.

Exercise 7.* Let $(Q, \mathfrak{m})$ be a regular local ring and F be a complex of finitely generated free Q-modules, not necessarily bounded on either side. Show that F is exact if and only if $F \otimes_Q Q/\mathfrak{m}$ is exact.

Exercise 8. Let Q be a regular local ring and $R = Q/I$ with I minimally generated by $f_1, \ldots, f_n$. Let F be a free resolution of R over Q that has a structure of a strictly graded commutative DG Q-algebra. Fix $e_1, \ldots, e_n \in F_1$ with $\partial(e_i) = f_i$. Show that we get a system of higher homotopies $\{\sigma_\omega\}$ for $\underline{f}$ on F by setting

$$\sigma_{\mathbf{e}_i}(-) = e_i \cdot - \qquad \text{and} \qquad \sigma_\omega(u) = 0 \text{ for all } |\omega| \geqslant 2.$$

[1] In lecture we showed these monomials form a spanning set.

Exercise 9.* Let $(Q, \mathfrak{m}, k)$ be a regular local ring and $R = Q/(f_1, \dots, f_c)$ for some $f_1, \dots, f_c \in \mathfrak{m}^2$. Show that under an appropriate choice of resolution and system of higher homotopies, the Shamash construction leads to the minimal free resolution for k over R.

Exercise 10.!! Let $R = \mathbb{F}_p[\![x_1, \dots, x_n]\!]/(x_1^p, \dots, x_n^p)$. Show that for each $0 \leqslant i \leqslant n$, there exists an R-module M such that $\mathrm{cx}_R(M) = i$.
Hint: As a first case, prove it for $n = 2$.

Exercise 11. Let $S = k[\chi_1, \dots, \chi_n]$ be a polynomial ring over a field k, and give S a positive grading, meaning that $\deg(\chi_i) \geqslant 1$ for each i. Prove that given a finitely generated graded R-module M,

$$M_d = 0 \text{ for all } d \gg 0 \iff \operatorname{Supp}(M) \subseteq \{(\chi_1, \dots, \chi_n)\}.$$

Exercise 12. Let M be a finitely generated R-module.

(a) Prove that $\mathrm{V}_R(M) = \mathrm{V}_R(\mathrm{Syz}_i^R(M))$ for any $i \geqslant 0$, whenever R is complete intersection.

(b) Does the equality in part (a) still hold when R is not complete intersection?

Exercise 13.* Let k be any field. Compute $\mathrm{V}_R(R)$ for $R = k[\![x, y]\!]/(xy, xz, yz)$.

Exercise 14. Compute $\mathrm{V}_R(R)$ where $R = k[\![x, y, z]\!]/(x^2, xy, z^2)$, where k is a field.
Hint: Use Exercise 5

Exercise 15.* Let $(R, \mathfrak{m}, k)$ be a local ring. In this problem you will prove that

$$\mathrm{thick}_{\mathsf{D}(R)}(k) = \{M \in \mathsf{D}(R) : \mathrm{length}_R \mathrm{H}(M) < \infty\}.$$

(a) Prove the containment "$\subseteq$" holds.

(b) If M has one nonzero homology module that is finite length, prove that M is in $\mathrm{thick}_{\mathsf{D}(R)}(k)$.

(c) For any R-complex M with $n = \max\{i : \mathrm{H}_i(M) \neq 0\}$ prove that there is an exact triangle

$$\Sigma^n \mathrm{H}_n(M) \to M \to M' \to$$

in $\mathsf{D}(R)$ where $\mathrm{H}_j(M') = \begin{cases} \mathrm{H}_j(M) & j < n \\ 0 & j \geqslant n\,. \end{cases}$

(d) Use parts (b) and (c) and induction to deduce the reverse containment "$\supseteq$" holds.