Singularities in Commutative Algebra Through Cohomological Methods

Notes for an SLMath Graduate Summer School

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July 5, 2026

Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or errors, we will be most grateful to you for letting us know.

Acknowledgements

These notes were prepared for a Graduate Summer School held at SLMath in July 2026. We are very thankful to SLMath for making this school possible. We also thank the teaching assistants for the school, Maya Banks and Ryan Watson, for their invaluable help.

These notes take much inspiration from other sources, especially Avramov's *Infinite free resolutions* [Avr10].

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Chapter 0

Background

0.1 Setup

Throughout, unless otherwise stated, noetherian local rings are assumed to be commutative with identity 1. Moreover, whenever we talk about modules over a noetherian local ring, we assume them to be unital, and all ring homomorphisms between noetherian local rings send 1 to 1. Noncommutative rings will also play an important role when we talk about dg algebras and universal enveloping algebras; the commutativity assumption only applies in the local case.

We will be primarily be concerned with noetherian local rings and finitely generated modules over such rings. However, the reader may be more familiar with the case of graded rings and modules. There are many parallels between these two settings, where the homogeneous maximal ideal in the graded case plays an analogous role to the unique maximal ideal in the local case. There is a sort of metatheorem that says that *most* theorems that hold in the local case have an analogous result in the graded setting, by adding in the words *homogeneous* and *graded* in the appropriate places. However, this does not mean that it is sufficient to prove a theorem in one of the two settings to obtain the other: often one needs to adapt the techniques slightly, and in rare occasions we even need an entirely different argument in each case. We should view this analogy as informing what we expect to be true in each case, but still treat the two cases as separate.

While these notes focus on the local case, many of the theorems we will state hold in both settings, though we note that the proofs can be quite different. The table below indicates the translations between the two settings that may serve as motivation to any reader who feels more comfortable in the graded setting.

local setting	graded setting
$(R, \mathfrak{m}, k)$ noetherian local ring	$R = k[x_1, \ldots, x_d]/I$ $k[x_1, \ldots, x_d]$ standard graded, k field, I homogeneous
M is a finitely generated R-module	M is a finitely generated graded R-module
$\mathfrak{m}$ the unique maximal ideal	$\mathfrak{m} = (x_1, \ldots, x_d)$ unique homogeneous maximal ideal

0.2 Free resolutions

Let R be a noetherian local ring. Given a finitely generated R-module M, we can completely describe M by giving a **presentation**, which consist of an R-module homomorphism

$$R^m \xrightarrow{\pi} R^n$$

with $\operatorname{coker}(\pi) = M$. If $x_1, \dots, x_n \in M$ are the images of a basis for R^n under the canonical quotient $R^n \twoheadrightarrow \operatorname{coker}(\pi) = M$, then $x_1, \dots, x_n$ form a generating set for M. Moreover, the images of a basis for R^m give generators for the **relations** among $x_1, \dots, x_n$. More precisely, $(a_1, \dots, a_n) \in \operatorname{im}(\pi)$ if and only if

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

Note that since M is finitely generated and R is noetherian, we can choose such a presentation with m and n finite. In our local setting, we may even take such a presentation to be minimal, meaning we can choose m and n to be smallest possible. This is equivalent to choosing $x_1, \dots, x_n$ such that their images in $M/\mathfrak{m}M$ form a basis for $M/\mathfrak{m}M$ as a vector space of $k = R/\mathfrak{m}$, and to choose π such that $\operatorname{im}(\pi) \subseteq \mathfrak{m}R^n$.

While such a presentation (minimal or not) determines M completely, we get a lot more information about M if we continue this process and construct a free resolution for M.

Definition 0.2.1. Let be M an R-module. A **free resolution** of M is a complex of free R-modules

$$\begin{array}{ccccccccccc} F = \cdots & \longrightarrow & F_n & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & 0 \\ & & \scriptstyle n & & & & \scriptstyle 1 & & \scriptstyle 0 & & \end{array}$$

together with an isomorphism $\mathrm{H}_0(F) \cong M$ such that $\mathrm{H}_i(F) = 0$ for all $i \neq 0$. We will abuse notation and carelessly identify a resolution for M with the **augmented resolution**

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & F_n & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0. \\ & & \scriptstyle n & & & & \scriptstyle 1 & & \scriptstyle 0 & & & & \end{array}$$

Remark 0.2.2. We can also talk about projective resolutions, where instead we allow the F_i to be projective modules. Over a noetherian local ring, Kaplansky showed that every projective module is free [Kap58], so we will focus only on free resolutions.

Every R-module has a free resolution; we will see next that this is a consequence of the fact that every R-module is a quotient of a free module. One indicates this by saying that the category of R-modules has enough projectives.

Construction 0.2.3 (Free resolution)**.** Let M be a finitely generated module over R with β_0 generators, say $m_1, \dots, m_{\beta_0}$. The R-module homomorphism induced by

$$\begin{array}{rcl} R^{\beta_0} & \xrightarrow{\pi_0} & M \\ \mathbf{e}_i & \longmapsto & m_i \end{array}$$

is surjective by construction.

If π_0 is an isomorphism, then $M \cong R^{\beta_0}$ is a **free module** of **free rank** β_0. Otherwise, π_0 has a nonzero kernel $\ker(\pi_0)$, which must also be a finitely generated module since R is noetherian. Fix generators $s_1, \dots, s_{\beta_1}$ for $\ker(\pi_0)$. We now consider the surjective R-module homomorphism induced by

$$\begin{aligned} R^{\beta_1} &\xrightarrow{\pi_1} \ker(\pi_0) \\ \mathbf{e}_i &\longmapsto s_i. \end{aligned}$$

Composing π_1 with the inclusion of $\ker(\pi_0)$ into R^{β_0}, we get a map $\partial_1 \colon R^{\beta_1} \longrightarrow R^{\beta_0}$; this is a presentation for M, and we get an exact sequence

$$\begin{tikzcd} R^{\beta_1} \arrow[rr, dashed, "\partial_1"] \arrow[dr, two heads, "\pi_1"'] & & R^{\beta_0} \arrow[r, two heads, "\pi_0"] & M. \\ & \ker(\pi_0) \arrow[ur, hook] & & \end{tikzcd}$$

We can then continue this process and construct a free resolution for M:

$$\cdots \longrightarrow F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0.$$

At each step, we find the kernel of $\partial_d \colon R^{\beta_d} \longrightarrow R^{\beta_{d-1}}$. If $\ker(\partial_d) = 0$, then the resolution has stopped:

$$0 \longrightarrow F_d \xrightarrow{\partial_d} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0.$$

Otherwise, we find a surjection from a free module $\pi_{d+1} \colon R^{\beta_{d+1}} \twoheadrightarrow \ker(\pi_d)$ and compose it with the inclusion $\ker(\pi_d) \hookrightarrow R^{\beta_d}$ to obtain the next map $\partial_{d+1} \colon R^{\beta_{d+1}} \longrightarrow R^{\beta_d}$.

At each step, as long as we are in the local setting, we can choose F_d to have the minimal number of generators possible; in that case, we say that F is a **minimal free resolution** for M. It turns out that no matter what choices we make along the way, the minimal number of generators for the free module in degree d is a well-defined invariant of M. In fact, the following hold:

- Every free resolution of M has a minimal free resolution of M as a direct summand.
- Any two minimal free resolutions of M are isomorphic complexes. From now on, we will refer to *the* minimal free resolution of M.
- As a consequence of the previous facts, the minimal free resolution of M must have the shortest length of any resolution for M, and thus M has a finite resolution if and only if the minimal free resolution of M is finite.
- A free resolution F of M with differential ∂ is minimal if and only if $\partial(F) \subseteq \mathfrak{m}F$. Thus if we fix bases for all the free modules F_i, the resolution is minimal if and only if all the entries in the matrices representing ∂ have all entries in $\mathfrak{m}$.

Definition 0.2.4 (Betti numbers)**.** Let F be the minimal free resolution of M. The i**th Betti number** of M is the free rank of F_i:

$$\beta_i(M) := \operatorname{rank}(F_i).$$

Remark 0.2.5. If F is the minimal free resolution for M, $F_i \cong R^{\beta_i(M)}$.

Exercise 0.2.6. Show that $\beta_i(M) = \dim_k \left(\operatorname{Tor}_i^R(M,k)\right) = \dim_k \left(\operatorname{Ext}_R^i(M,k)\right)$.

Remark 0.2.7. Note that in particular $\beta_0(M) = \mu(M) = \dim_k(M/\mathfrak{m}M)$ is the minimal number of generators of M.

Such relations are called **syzygies**.

Definition 0.2.8. Consider a minimal free resolution F of M. The n**th syzygy module of** M,[1] denoted $\operatorname{Syz}_n(M)$, is defined to be the image of ∂_n, or equivalently the kernel of ∂_{n-1}.

Definition 0.2.9. Let M be a nonzero R-module. A finite free resolution

$$F = \cdots \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

has length c if $F_c \neq 0$ and $F_i = 0$ for all $i \geqslant c$. A free resolution F has infinite length if $F_i \neq 0$ for all $i \geqslant 0$. The **projective dimension** of M is

$$\begin{aligned} \operatorname{pdim}_R(M) &:= \inf\{c \mid M \text{ has a projective resolution of length } c\} \\ &= \text{length of any minimal free resolution for } M. \end{aligned}$$

We may write $\operatorname{pdim}(M)$ whenever the underlying ring is clear from context.

Macaulay2. In `Macaulay2`, we can compute free resolutions using the command `res`. For an R-module M, `res(M)` will return a free resolution for M, which will by default be minimal as long as M is a graded R-module. Given an ideal I, `res(I)` outputs a resolution for R/I.

The projective dimension of a finitely generated module can be both finite and infinite.

Example 0.2.10. Let k be a field, $R = k[\![x]\!]$, and consider the residue field $k = R/(x)$. This R-module is cyclic, and the kernel of the canonical map $R \twoheadrightarrow R/(x)$ has only one generator, (x), so the resolution begins with

$$R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0.$$

Since R is a domain, x is a regular element, that is, not a zerodivisor, and thus the leftmost map is injective, completing our resolution. Since x is in the maximal ideal, we conclude that this is the minimal free resolution for k:

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0.$$

[1]Fun fact: in astronomy, a syzygy is an alignment of three or more celestial objects.

Example 0.2.11. Let k be a field and $R = k[\![x]\!]/(x^3)$. When we resolve the residue field $k \cong R/(x)$, we now note that multiplication by x on R does in fact have a nontrivial kernel, and thus the resolution is a bit more complicated. In fact, the resolution is infinite:

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x^2} R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0.$$

In particular, $\operatorname{pdim}_R(k) = \infty$ and $\beta_i(k) = 1$ for all i.

The last two examples illustrate a very important theorem of Auslander–Buchsbaum and Serre. First, it is not a coincidence that the projective dimension of the residue field is finite in the first case and infinite in the second case: this is a reflection of the fact that $k[\![x]\!]$ is a regular ring while $k[\![x]\!]/(x^3)$ is singular. In fact, the residue field contains a lot of information.

Exercise 0.2.12. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Show that for all finitely generated R-modules M,

$$\operatorname{pdim}_R(M) \leqslant \operatorname{pdim}_R(k).$$

Exercise 0.2.13. Let $(R, \mathfrak{m})$ be a noetherian local ring and A and B be R-modules such that A is a direct summand of B. Show that $\operatorname{pdim}_R(A) \leqslant \operatorname{pdim}_R(B)$.

What can resolutions tell us about singularities? First, we recall the definition of a regular ring; these are the rings that are nonsingular.

Definition 0.2.14. A noetherian local ring $(R, \mathfrak{m}, k)$ is a **regular local ring** if $\mathfrak{m}$ is minimally generated by $\dim(R)$ many elements, that is

$$\mu(\mathfrak{m}) = \dim(R).$$

The minimal number of generators of $\mathfrak{m}$ is known as the **embedding dimension** of R, written $\operatorname{embdim}(R)$. We will sometimes write **RLR** as shorthand for regular local ring.

Remark 0.2.15. Recall that by Krull's Height Theorem, we always have

$$\dim(R) = \operatorname{height}(\mathfrak{m}) \leqslant \mu(\mathfrak{m}) = \operatorname{embdim}(R).$$

Thus a local ring is regular if its maximal ideal has the smallest possible number of generators.

I. S. Cohen [Coh46] proved that any complete regular local ring containing a field is of the form $k[\![x_1, \ldots, x_d]\!]$ for k a field. In fact, he proved more:

Theorem 0.2.16 (Cohen's Structure Theorem)**.** *Every complete noetherian local ring can be written as a quotient of a regular local ring.*

Exercise 0.2.17. Let $(R, \mathfrak{m}, k)$ be a regular local ring. Show that $R/(f)$ is a regular local ring if and only if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.

Thanks to Cohen's Structure Theorem and Exercise 0.2.17, we can write any complete regular ring in the form $R = Q/I$ with $(Q, \mathfrak{m})$ a regular local ring and $I \subseteq \mathfrak{m}^2$.

Definition 0.2.18. Given a noetherian local ring R, a **minimal regular presentation** for R consists of a regular local ring $(Q, \mathfrak{m})$ and an ideal $I \subseteq \mathfrak{m}^2$ in Q such that $\widehat{R} \cong Q/I$, where $\widehat{R}$ is the completion of R at its maximal ideal.

Exercise 0.2.19. Show that if $\widehat{R} \cong Q/I$ is a minimal regular presentation for R, then

$$\operatorname{embdim}(Q) = \operatorname{embdim}(\widehat{R}) = \operatorname{embdim}(R).$$

For those unfamiliar with completions or DVRs, a good starting point is to assume that from now on we will be talking about complete equicharacteristic rings: rings of the form $R = k[\![x_1, \dots, x_e]\!]/I$, where k is a field and $I \subseteq (x_1, \dots, x_e)^2$.

Note that in such a minimal regular presentation, $I = 0$ if and only if R is a regular ring. When R is not regular, we will say R is a **singular ring**. The big goal of these lectures is to introduce homological tools that can be used to detect singularities. The next theorem is the first instance of this phenomenon.

Theorem 0.2.20 (Auslander–Buchsbaum, Serre [AB57, Ser56])**.** *The following are equivalent for any noetherian local ring $(R, \mathfrak{m}, k)$:*

1) The ring R is regular.

2) The residue field k has finite projective dimension.

3) Every finitely generated R-module has finite projective dimension.

This theorem marks the advent of homological methods in commutative algebra. One amazing consequence of having such a homological characterization of regular rings is that it allows us to solve the Localization Problem for regular rings: if P is a prime ideal in a regular local ring R, is R_P a regular local ring? Many good properties of a local ring are preserved by localization, so this is a reasonable question. Note that a positive answer to the Localization Problem gives us a natural choice for a global definition of regular ring: a noetherian ring R is regular if R_P is a regular local ring for all primes P.

This problem is unwieldy via the embedding dimension definition, as it would require us to show that for all primes P,

$$\mu(P_P) = \dim(R_P) = \operatorname{height}(P).$$

But this now follows easily via the homological characterization of regularity.

Exercise 0.2.21 (The Localization Problem for Regular Rings)**.** Let R be a regular local ring. Show that for all prime ideals P, the localization R_P is a regular local ring.

One big difference between Example 0.2.10 and Example 0.2.11 is whether x is a regular element on R. This leads us to the next topic: regular sequences.

0.3 Regular sequences and depth

While we assume that the reader is familiar with regular sequences and depth, we will now collect the most important definitions and facts we will use throughout these lectures. More details on these topics can be found in Bruns and Herzog's *Cohen-Macaulay rings* [BH98].

Definition 0.3.1 (Regular sequence)**.** Let R be a ring and M be an R-module. An element $x \in R$ is **regular** on M if $xM \neq M$ and for any $m \in M$

$$xm = 0 \Rightarrow m = 0.$$

Definition 0.3.2. A sequence of elements $x_1, \dots, x_n$ is a **regular sequence on** M if

- $(x_1, \dots, x_n)M \neq M$, and
- for each i, the element x_i is regular on $M/(x_1, \dots, x_{i-1})M$.

When $M = R$, we may drop the *on* M and say $x_1, \dots, x_n$ is a regular sequence.

In general, whether $x_1, \dots, x_n$ is a regular sequence depends on the order of the elements. However, in the local setting, this is independent of the order of the elements. One can also show that all maximal regular sequences on M have the same length.

Definition 0.3.3. Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. The maximal length n of $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$ a regular sequence on M is the **depth** of M, denoted $\operatorname{depth}(M)$.

Exercise 0.3.4. Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. Show that $\operatorname{depth}(M) = 0$ if and only if $\mathfrak{m} \in \operatorname{Ass}(M)$.

Finding a maximal regular sequence on a module M boils down to identifying associated primes of M and some of its quotients.

Remark 0.3.5. Here is a recipe for finding the depth of M, and for constructing an explicit maximal regular sequence on M.

First, we determine $\operatorname{Ass}(M)$. If $\mathfrak{m} \in \operatorname{Ass}(M)$, then $\operatorname{depth}(M) = 0$ and we are done. If not, then we find an element $x_1 \in \mathfrak{m}$ that is not in any associated prime of M, which exists by Prime Avoidance. Such an x_1 is necessarily regular on M, given that the union of the associated primes of M coincides with the set of zerodivisors on M.

We then proceed analogously, now on the module $M/(x_1)M$. At each step, we determine whether $\mathfrak{m}$ is associated to $M/(x_1, \dots, x_i)M$, and if not then we find a new element $x_{i+1} \in \mathfrak{m}$ that is not in any of the associated primes of $M/(x_1, \dots, x_i)M$. At each stage, the depth drops by one, and thus this process must eventually stop.

We can also measure depth via the (non)vanishing certain Ext modules.

Theorem 0.3.6. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated R-module. Then*

$$\operatorname{depth}(M) = \min\{i \mid \operatorname{Ext}^i_R(k, M) \neq 0\}.$$

Theorem 0.3.7 (Auslander–Buchsbaum Formula)**.** *Let R be local and let M be a finitely generated (graded) R-module of finite projective dimension. Then*

$$\operatorname{pdim}_R(M) + \operatorname{depth}(M) = \operatorname{depth}(R).$$

Given a noetherian local ring R, one can show that we always have

$$\operatorname{depth}(R) \leqslant \dim(R).$$

It is natural to ask when equality holds; this leads to a very important class of rings.

Definition 0.3.8. A noetherian local ring R is **Cohen-Macaulay** if $\operatorname{depth}(R) = \dim(R)$.

Mel Hochster famously wrote that *life is really worth living in a (...) Cohen-Macaulay ring.*[2] Hochster was likely referring to the vast wealth of nice properties that Cohen-Macaulay rings enjoy. These are often properties one would hope are true in general, but that (sometimes surprisingly) can fail for rings that are not Cohen-Macaulay. For example, over a Cohen-Macaulay ring we have the following desirable dimension formula:

Theorem 0.3.9. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, and let I be any proper ideal in R. Then*

$$\dim(R/I) = \dim(R) - \operatorname{height}(I).$$

For our purposes, we will often use this property in the special case when R is a regular local ring; as a consequence of Theorem 0.3.12, all regular local rings are Cohen-Macaulay.

Over any Cohen-Macaulay ring (so for example, over any regular ring), we have a nice simple characterization of ideals generated by a regular sequence:

Corollary 0.3.10. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, and let I be a proper ideal in R. The following are equivalent:*

1) The ideal I is generated by a regular sequence.

2) Every minimal generating set for I is a regular sequence.

3) $\mu(I) = \operatorname{height}(I)$.

Remark 0.3.11. Recall that by Krull's Height Theorem, we always have

$$\mu(I) \leqslant \operatorname{height}(I).$$

Thus an ideal is generated by a regular sequence exactly when it has the smallest possible number of generators for its height.

This should remind us of Remark 0.2.15. Indeed:

Theorem 0.3.12. *A noetherian local ring $(R, \mathfrak{m})$ is regular if and only if $\mathfrak{m}$ is generated by a regular sequence.*

[2] "Life is really worth living in a noetherian ring R when all the local rings have the property that every system of parameters is an R-sequence. Such a ring is called Cohen-Macaulay (C-M for short)."

Chapter 1

Differential graded algebras

1.1 The Koszul complex

Definition 1.1.1 (Koszul complex)**.** The **Koszul complex** on $x_1, \dots, x_n \in R$, denoted $\mathrm{Kos}^R(x_1, \dots, x_n)$, or $\mathrm{Kos}(x_1, \dots, x_n)$ if R is clear from context, is a complex of free R-modules, defined as follows.

- In degree d, we have the R-module
$$\wedge^d(R^n) = R^{\binom{n}{d}}$$
with basis
$$e_{i_1} \wedge \cdots \wedge e_{i_d} \qquad \text{where } 1 \leqslant i_1 < \cdots < i_d \leqslant n.$$
In particular, note that $\mathrm{Kos}(x_1, \dots, x_n)$ lives in homological degrees 0 through n.

- The differential is defined on basis elements as follows:
$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{1 \leqslant p \leqslant s} (-1)^{p-1} x_{i_p} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s}.$$

Example 1.1.2. In the case of two elements, say f and g in R,

$$\mathrm{Kos}(x, y) = \ 0 \longrightarrow \wedge^2 R^2 \longrightarrow \wedge^1 R^2 \longrightarrow R \longrightarrow 0$$

with $\partial(e_1) = f$, $\partial(e_2) = g$, $\partial(e_1 \wedge e_2) = fe_2 - ge_1$, so

$$\mathrm{Kos}(x, y) = \ 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -g \\ f \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} R \longrightarrow 0.$$

Exercise 1.1.3. Write the Koszul complex on 3 elements f_1, f_2, f_3.

Note, however, that the Koszul complex may have nontrivial homology.

Theorem 1.1.4. *Let R be local, and let $f_1, \dots, f_n \in R$ be nonunits. The following are equivalent:*

1) *The elements $f_1, \dots, f_n$ form a regular sequence.*

2) *The Koszul complex $\operatorname{Kos}(f_1, \dots, f_n)$ is a free resolution of $R/(f_1, \dots, f_n)$, that is, $\mathrm{H}_i(\operatorname{Kos}(f_1, \dots, f_n)) = 0$ for all $i \neq 0$.*

3) *The first Koszul homology vanishes: $\mathrm{H}_1(\operatorname{Kos}(f_1, \dots, f_n)) = 0$.*

Remark 1.1.5. The nonzero entries in the differentials of $\operatorname{Kos}(\underline{f})$ are all of the form $\pm f_i$, and thus when $\underline{f}$ is a regular sequence $\operatorname{Kos}(\underline{f})$ is a minimal free resolution for $R/(\underline{f})$.

Remark 1.1.6. By Theorem 0.3.12, when R is regular, the maximal ideal is generated by a regular sequence. The Koszul complex on such a regular sequence is thus a minimal free resolution for k over R.

One can also define the Koszul complex inductively, by taking successive tensor products.

Notation 1.1.7. Given any complex A, we write $|a| := i$ to indicate that $a \in A_i$.

Definition 1.1.8. Let A and B be complexes of R-modules. The **tensor product** of A and B is the complex $A \otimes_R B$ with

$$(A \otimes_R B)_n = \bigoplus_{i+j=n} A_i \otimes_R B_j$$

and

$$\partial_{A \otimes_R B}(a \otimes b) = \partial_A(a) \otimes b + (-1)^{|a|} a \otimes \partial_B(b).$$

Remark 1.1.9. The tensor product of two complexes A and B is the totalization of the double complex with

$$(A \otimes B)_{p,q} = A_p \otimes B_q \quad d^h = \partial_A \otimes \mathrm{id}_B, \quad \text{and} \quad d^v = (-1)^p \,\mathrm{id}_A \otimes \partial_B.$$

We are now ready for an alternative definition of the Koszul complex.

Construction 1.1.10 (The Koszul complex)**.** The **Koszul complex** on one element $x \in R$ is the complex

$$\operatorname{Kos}(x) := 0 \longrightarrow \underset{1}{R} \xrightarrow{\;x\;} \underset{0}{R} \longrightarrow 0.$$

More generally, given $\underline{x} = x_1, \dots, x_n \in R$, the **Koszul complex** with respect to $\underline{x}$ is the complex $\operatorname{Kos}^R(\underline{x}) = \operatorname{Kos}(x_1, \dots, x_n)$ defined inductively as

$$\operatorname{Kos}(x_1, \dots, x_n) := \operatorname{Kos}(x_1, \dots, x_{n-1}) \otimes_R \operatorname{Kos}(x_n).$$

Example 1.1.11. The Koszul complex on $f, g \in R$ is given by

$$\operatorname{Kos}(f,g) = \text{Totalization of} \quad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & R \arrow[r, "f"] \arrow[d, "-g"'] & R \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & R \arrow[r, "f"'] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & \end{tikzcd}$$

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which is

$$0 \longrightarrow R \xrightarrow{\binom{-g}{f}} R^2 \xrightarrow{(f \quad g)} R \longrightarrow 0.$$

Exercise 1.1.12. Let $I = (x_1, \dots, x_n)$ be an ideal in a noetherian ring R, and $\underline{x} = x_1, \dots, x_n$.

1) $\mathrm{H}_i(\operatorname{Kos}(\underline{x})) = 0$ whenever $i < 0$ or $i > n$.
2) $\mathrm{H}_0(\operatorname{Kos}(\underline{x})) = R/I$.
3) $\mathrm{H}_n(\operatorname{Kos}(\underline{x})) = (0 : I) = \operatorname{ann}(I)$.
4) The Koszul homology modules $\mathrm{H}_i(\operatorname{Kos}(\underline{x}))$ are finitely generated for every i.
5) Every Koszul homology module $\mathrm{H}_i(\operatorname{Kos}(\underline{x}))$ is killed by I.
   Hint: show that for all $a \in I$, multiplication by a on $\operatorname{Kos}(\underline{x})$ is nullhomotopic.

Remark 1.1.13. Set $C := \operatorname{Kos}(x_1, \dots, x_n)$, and note that

$$[\operatorname{Kos}(x_1, \dots, x_{n+1})]_i = C_{i-1} \otimes_R R \oplus C_i \otimes_R R \cong C_{i-1} \oplus C_i.$$

Given $a \in C_{i-1}$, $b \in C_i$, $r \in R$ (in homological degree 1) and $s \in R$ (in homological degree 0),

$$\partial(a \otimes r + b \otimes s) = \partial(a) \otimes r + (-1)^{i-1} a \otimes (x_{n+1} r) + \partial(b) \otimes s + (-1)^i b \otimes 0,$$

so

$$\partial_i := \begin{pmatrix} \partial_C & 0 \\ (-1)^{i-1} x_{n+1} & \partial_C \end{pmatrix} : \quad \begin{tikzcd} C_{i-1} \arrow[r, "\partial_C"] \arrow[ddr, "(-1)^{i-1}x_{n+1}"] & C_{i-2} \\ \oplus & \oplus \\ C_i \arrow[r, "\partial_C"'] & C_{i-1}. \end{tikzcd}$$

In fact, this is the cone of the map $\operatorname{Kos}(x_1, \dots, x_n) \longrightarrow \operatorname{Kos}(x_1, \dots, x_n)$ given by multiplication by x_{n+1} in every degree. Note that there are different sign conventions on what the differential of a cone is, so this might not match your favorite definition. For the reader unfamiliar with cones, we will discuss this topic in more detail later.

1.2 Differential graded algebras

When we first introduced the Koszul complex, we used the suggestive notation $\wedge^d(R^n)$ when talking about the free modules in each homological degree. Indeed, the Koszul complex has more structure than simply being a complex: it is the exterior algebra on n elements. In fact, the Koszul complex is an example of a differential graded algebra, which we will abbreviate to dg algebra.

Definition 1.2.1. Let R be a commutative ring. A dg (differential graded) algebra over R is a complex (A, ∂) of R-modules equipped with an associated algebra structure compatible with the differential, as follows:

1) The underlying graded object
$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$
is a unital associative graded R-algebra.

2) The differential ∂ satisfies the **Leibniz rule**: for all $a, b \in A$, we have
$$\partial(ab) = \partial(a)b + (-1)^{|a|} a\partial(b).$$
Therefore, the multiplication $A \otimes_R A \longrightarrow A$ is a map of complexes.

Definition 1.2.2. A dg algebra is a **(strictly) graded commutative dg algebra** if for all homogeneous elements a and b,
$$ab = (-1)^{|a| \cdot |b|} ba \qquad \text{and} \qquad a^2 = 0 \quad \text{whenever } a \text{ has odd degree.}$$

Remark 1.2.3. Note that in characteristic other than 2, the condition
$$ab = (-1)^{|a| \cdot |b|} ba$$
automatically implies that $a^2 = 0$ whenever $|a|$ is odd. However, this does not follow in characteristic 2, and thus we add this condition as well. The *stricly* in strictly graded commutative refers to this added condition that $a^2 = 0$ for all $|a|$ odd. While some authors refer to graded commutative and strictly graded commutative dg algebras separately, we will not consider one condition without the other, so we may shorten it graded commutative.

Remark 1.2.4. Note that the fact that $A_0 A_0 \subseteq A_0$ implies that A_0 is a (noncommutative) ring. Moreover, A_0 is a commutative ring if A is graded commutative.

Example 1.2.5. Given a ring R, the complex with R in degree 0 and zeroes elsewhere is a dg R-algebra. More generally, an R-algebra is exactly a dg R-algebra concentrated in degree 0.

The canonical example of a dg algebra is the Koszul complex.

Example 1.2.6 (the Koszul complex revisited)**.** Given elements $\underline{f} = f_1, \dots, f_n$ in a commutative ring R, the Koszul complex $E = \mathrm{Kos}(\underline{f})$ is a graded commutative R-algebra with the wedge product induced by

$$\left(e_{i_1} \wedge \cdots \wedge e_{i_s}\right) \cdot \left(e_{j_1} \wedge \cdots \wedge e_{j_t}\right) = e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}.$$

Note that $\mathrm{Kos}(\underline{f})$ is the exterior algebra on $e_1, \dots, e_n$. The differential in Definition 1.1.1 is the unique differential with $\partial(e_i) = f_i$ that satisfies the Leibniz rule with respect to this product. We also use the alternative notation

$$\mathrm{Kos}(\underline{f}) = R[e_1, \dots, e_n \mid \partial(e_i) = f_i],$$

which indicates that the Koszul complex was obtained from R by *adjoining* exterior variables $e_1, \dots, e_n$ that kill the cycles $f_1, \dots, f_n$ in degree 0 into boundaries, giving us a dg algebra with

$$\mathrm{H}_0(R[e_1, \dots, e_n \mid \partial(e_i) = f_i]) = R/(f_1, \dots, f_n).$$

Notation 1.2.7. Given a complex, we write $[x]$ to denote the homology class of a cycle x.

Remark 1.2.8. Let A be a dg R-algebra. Note that the product of two cycles is a cycle, and thus the cycles $\mathrm{Z}(A)$ of A form a subalgebra of A. Moreover, the boundaries $\mathrm{B}(A)$ of A form an ideal of $\mathrm{Z}(A)$. We conclude that the homology of A

$$\mathrm{H}(A) = \bigoplus_i \mathrm{H}_i(A)$$

forms a graded R-algebra with multiplication induced by the multiplication on A, meaning that

$$[a] \cdot [b] := [a \cdot b].$$

We leave the details as an exercise (see below). Note that if A is graded commutative, then so is $\mathrm{H}(A)$.

Exercise 1.2.9. For a dg algebra A, prove that the multiplication $[a] \cdot [b] := [a \cdot b]$ is well-defined and makes $\mathrm{H}(A)$ a graded algebra.

Definition 1.2.10. Let A and B be dg algebras over R. A **homomorphism of dg algebras** $f\colon A \longrightarrow B$ is a (degree 0) map of complexes such that

$$f(xy) = f(x)f(y) \qquad \text{and} \qquad f(1_A) = 1_B.$$

A map of dg algebras is an isomorphism whenever the underlying map of complexes is an isomorphism. Moreover, $f\colon A \longrightarrow B$ is a **quasiisomorphism of dg algebras** if f is a map of dg algebras and $\mathrm{H}(f)\colon \mathrm{H}(A) \longrightarrow \mathrm{H}(B)$ is an isomorphism.

Remark 1.2.11. Let F be a free resolution for an R-module M, and consider the canonical surjection $\pi\colon F_0 \longrightarrow \mathrm{H}_0(F) \cong M$. We can view M as a complex concentrated in homological degree 0, and extend π to a map of complexes $F \longrightarrow M$, which is a quasiisomorphism of complexes. We will often denote this by

$$F \xrightarrow{\simeq} M$$

where the symbol $\simeq$ denotes a quasiisomorphism (of either complexes or dg algebras, according to context). If $M = R/I$ and F has a dg algebra structure, note that this is a quasiisomorphism of dg algebras.

Exercise 1.2.12. Let A and B be dg algebras over R. Check that the tensor product complex $A \otimes_R B$ is also a dg algebra, under the product induced by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2|\cdot|b_1|}(a_1 a_2) \otimes (b_1 b_2).$$

Moreover, show that

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes_R B & \longleftarrow & B \\ a & \longmapsto & a \otimes 1 & & \\ & & 1 \otimes b & \longleftarrow\!\shortmid & b \end{array}$$

are dg algebra maps.

Definition 1.2.13. Given a dg algebra A, an ideal I of the (noncommutative) ring A is a **dg ideal** of A if $\partial_A(I) \subseteq I$.

Example 1.2.14. Let $\underline{f} \in R$ and consider the Koszul complex $A = \mathrm{Kos}^Q(\underline{f})$. Note that the ideal $I = A_{\geqslant 1}$, which as an A-ideal is generated by $e_1, \ldots, e_n$, is not a dg ideal, since $\partial(e_i) = f_i \notin I$. However, the A-ideal

$$J = (e_1, \ldots, e_n, f_1, \ldots, f_n)$$

is a dg ideal. In fact, J is the dg ideal generated by $e_1, \ldots, e_n$.

Remark 1.2.15. Suppose that A is a dg algebra and I is a dg ideal in A. Then A/I is a dg algebra, and the canonical quotient map $A \twoheadrightarrow A/I$ is a dg algebra map.

Exercise 1.2.16. Let I be a dg ideal of the dg algebra A. Show that the canonical map $A \twoheadrightarrow A/I$ is a quasiisomorphism if and only if $\mathrm{H}(I) = 0$.

Definition 1.2.17. A dg algebra A is a **local dg algebra** if A has a unique maximal dg ideal. We may write $\mathfrak{m}_A$ to indicate the unique maximal dg ideal.

Example 1.2.18. Let $(R, \mathfrak{m})$ be a local ring and let $\underline{f} \in R$. The Koszul complex $A = \mathrm{Kos}(\underline{f})$ is a local dg algebra with unique maximal dg ideal

$$\mathfrak{m}_R \oplus \left(\bigoplus_{i=1}^{n} Re_i\right) \oplus \left(\bigoplus_{i<j} Re_ie_j\right) \oplus \cdots \oplus Re_1 \cdots e_n = \mathfrak{m}_R \oplus A_{\geqslant 1}.$$

Remark 1.2.19. More generally, let A be a nonnegatively graded dg algebra, meaning that $A_i = 0$ for all $i < 0$. If $A_0 = R$ is a local ring, then A is a local dg algebra with unique maximal dg ideal

$$\mathfrak{m}_A = \mathfrak{m}_R \oplus A_{\geqslant 1}.$$

Definition 1.2.20. A map of local dg algebras $\varphi\colon A \longrightarrow B$ is a **local map of dg algebras** if $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

Example 1.2.21. Let F and G be complexes of R-modules, and let $\operatorname{Hom}_R(F, G)$ denote the complex with

$$\operatorname{Hom}_R(F, G)_n := \prod_i \operatorname{Hom}_R(F_i, G_{i+n})$$

and differential

$$\partial(\varphi) = \partial_G \circ \varphi - (-1)^{|\varphi|} \varphi \circ \partial_F.$$

We leave it as an exercise to check that this is indeed a complex of R-modules; in fact, when $F = G$, this is a dg R-algebra with the product given by composition.

Exercise 1.2.22. Let M and N be R-complexes.

1) Check that $\operatorname{Hom}_R(M, N)$ is a complex.

2) Describe $\mathrm{Z}_0(\operatorname{Hom}_R(M, N))$ and $\mathrm{H}_0(\operatorname{Hom}_R(M, N))$.

3) Set $\operatorname{End}_R(M) = \operatorname{Hom}_R(M, M)$. Prove that $\operatorname{End}_R(M)$ is a dg R-algebra.

Example 1.2.23. Let F be any complex. Then $\operatorname{End}_R(F) = \operatorname{Hom}_R(F, F)$ is a dg algebra, though it is not graded commutative and not local.

Definition 1.2.24. A (left) **dg module** M over a dg R-algebra A is a complex of R-modules with the structure of a graded module over the underlying ring A that satisfies the Leibniz rule, that is, for all $a \in A$ and $m \in M$,

$$\partial^M(am) = \partial^A(a)m + (-1)^{|a|} a \partial^M(m).$$

Remark 1.2.25. Note that a dg module structure over a dg algebra A on a complex M is equivalent to specifying a map of complexes

$$\begin{aligned} A \otimes_R M &\longrightarrow M \\ a \otimes m &\longmapsto am \end{aligned}$$

that is unital (ie, that sends $1 \otimes m \mapsto m$) and associative.

Example 1.2.26. Given any dg algebra A, if we forget the differential, any dg module over A is in particular a left A-module.

Notation 1.2.27. The **suspension** or **shift** of a complex C is the complex ΣC, sometimes also denoted $C[1]$, with

$$(\Sigma C)_n = C_{n-1} \quad \text{and} \quad \partial^{\Sigma C} = -\partial^C.$$

More generally, for any ℓ, the **ℓth suspension** of C is the complex $\Sigma^\ell C$ with

$$(\Sigma^\ell C)_n := C_{n-\ell} \qquad \text{with } \partial^{\Sigma^\ell C} = (-1)^\ell \partial^C.$$

In particular, $\Sigma^{-1} C$ is the complex with

$$(\Sigma^{-1} C)_n = C_{n+1} \quad \text{and} \quad \partial^{\Sigma^{-1} C} = -\partial^C.$$

Remark 1.2.28. Note that

$$\Sigma^\ell C := \underbrace{\Sigma \cdots \Sigma}_{\ell \text{ times}} C.$$

Example 1.2.29. If A is a dg algebra, then ΣA is a dg A-module via

$$a \cdot (\Sigma b) := (-1)^{|a|} \Sigma(ab).$$

More generally, given an arbitrary dg A-module M, $\Sigma^\ell M$ is a dg A-module:

Exercise 1.2.30. Let A be a dg algebra and M a (left) dg A-module. Prove that for each ℓ, the complex $\Sigma^\ell M$ has a (left) dg A-module structure where the A-action is given by

$$a \cdot \Sigma^\ell m := (-1)^{|a|\ell} \Sigma^\ell (am)\,.$$

Example 1.2.31. Let F and G be complexes of R-modules. Then $\operatorname{Hom}_R(F, G)$ is a left dg $\operatorname{End}_R(G)$-module and a right dg $\operatorname{End}_R(F)$-module in a compatible way (i.e., a dg bimodule).

Example 1.2.32. If $\varphi\colon A \longrightarrow B$ is a map of dg algebras, then it endows B with the structure of a dg A-module, akin to restricting scalars along an ordinary ring homomorphism.

1.3 Adjoining variables to kill cycles

From now on, we will often have not one but two rings of interest: a noetherian local ring Q, and a quotient ring $R = Q/I$. We will often reserve Q for the case when Q is a regular local ring, though for now Q is only assumed to be a noetherian local ring.

When we build the Koszul complex on $f_1, \dots, f_n$, we adjoin n exterior variables $e_1, \dots, e_n$ to kill the cycles $f_1, \dots, f_n$, and the only relations we impose on these variables are the ones that are forced by the definition of a strict graded commutative algebra. Indeed, the Koszul $\operatorname{Kos}(\underline{f}) = \wedge_R(e_1, \dots, e_n)$ is the *free* strictly graded commutative dg algebra generated by $e_1, \dots, e_n$. We will now discuss a a higher order version of this idea: at each stage, we impose only the minimal relations required among our new variables.

Construction 1.3.1. In general, given a dg algebra A and a variable x, we write $A[x]$ to denote the free strictly graded commutative A-algebra on x, which is given by

$$A[x] = \begin{cases} A \oplus Ax & \text{the exterior algebra on } x \text{, if } |x| \text{ is odd} \\ A \oplus Ax \oplus Ax^2 \oplus \cdots & \text{the polynomial algebra on } x \text{, if } |x| \text{ is even.} \end{cases}$$

Exercise 1.3.2 (Killing cycles)**.** Let A be a dg algebra and $z \in A_n$ a cycle. Let x be a variable with $|x| = |z| + 1 = n + 1$. Show that there exists a unique strictly graded commutative dg algebra structure on $A[x]$ extending the differential in A and such that $\partial(x) = z$. Moreover,

$$\mathrm{H}_i(A[x]) = \begin{cases} \mathrm{H}_i(A) & \text{if } i < n \\ \mathrm{H}_n(A)/([z]) & \text{if } i = n. \end{cases}$$

Note that Exercise 1.3.2 says nothing about $\mathrm{H}_{>n}(A[x])$. Since

$$\mathrm{H}_n(A[x]) = \mathrm{H}_n(A)/([z]),$$

we say the cycle z is *killed*.

Inspired by ideas from topology, Tate [Tat57] constructed dg algebra resolutions over local rings, using this idea of adjoining variables to kill cycles.

Construction 1.3.3. Let Q be a noetherian local ring and $R = Q/I$. We will construct a dg algebra resolution for R in steps, by successively adding variables in each degree to kill homology in the degree below.

Step 0: Consider the complex Q concentrated in degree 0. Our goal is to change the homology of this complex so that it becomes R in degree 0, and zeroes elsewhere.

Step 1: Fix a generating set $f_1, \dots, f_n$ for I and adjoin variables $x_1, \dots, x_n$ of homological degree 1 so that $\partial(x_i) = f_i$. We write

$$Q[x_1, \dots, x_n \mid \partial(x_i) = f_i]$$

to represent the resulting complex, or $Q[X_1]$ with $X_1 = \{x_1, \dots, x_n\}$ for short.

The resulting complex starts with

$$\begin{array}{ccc} \displaystyle\bigoplus_{i=1}^{n} Q \cdot x_i & \xrightarrow{\partial} & Q \\ {\scriptstyle 1} & & {\scriptstyle 0} \end{array}$$

just as we would normally start with when building a resolution for R over Q, but these x_i are elements in a dg algebra, so we need to consider their products as well, which live in higher degrees. We take these to be exterior variables, so that the only relations among them are the ones necessary to satisfy the definition of a graded commutative dg algebra: we set

$$x_i x_j = -x_j x_i \quad \text{and} \quad x_i^2 = 0.$$

The differential on any other element of $Q[X_1]$ is now completely determined by linearity and the Leibniz rule. In fact, $Q[X_1]$ is simply the Koszul complex on $f_1, \dots, f_n$:

$$0 \longrightarrow Q \cdot x_1 \cdots x_n \longrightarrow \cdots \longrightarrow \bigoplus_{i<j} Q \cdot x_i x_j \longrightarrow \bigoplus_{i=1}^{n} Q \cdot x_i \xrightarrow{\partial} Q.$$

So far, we have managed to fix the homology in degree 0 to be R. If $\mathrm{H}_1(Q[X_1]) = 0$, then in fact by Theorem 1.1.4 the Koszul complex must be exact, and we have finished constructing a resolution for R. Otherwise, we proceed to step 2.

Before we proceed, note that since R is a noetherian ring and we are dealing with finitely generated modules, the R-module $\mathrm{H}_1(Q[X_1])$ is finitely generated.

Step 2: Fix cycles $z_1, \dots, z_s \in Q[X_1]$ of degree 1 whose homology classes $[z_1], \dots, [z_s]$ generate $\mathrm{H}_1(Q[X_1])$, and adjoin variables $x_{n+1}, \dots, x_{n+s}$ of degree 2 to kill the homology of degree 1, meaning that we set

$$\partial(x_{n+i}) = z_i.$$

We may take these variables of degree 2 to be of one of two kinds: polynomial variables or divided power variables. The divided power variable version of this construction is due to Tate [Tat57].

Let us first describe what happens when we take polynomial variables. In this case, there are no additional relations except for the fact that any two variables of degree 2 commute with each other and with all variables of degree 1. The differential of the resulting complex is completely determined by Q-linearity and the Leibniz rule.

Setting $X_2 = \{x_{n_1}, \dots, x_{n+s}\}$, we have

$$\mathrm{H}_0(Q[X_1, X_2]) = R \quad \text{and} \quad \mathrm{H}_1(Q[X_1, X_2]) = 0.$$

We continue in this fashion, repeating this process in every degree. Note once more that the fact that R is noetherian and that at each stage we add finitely many variables guarantees that the successive homology modules $\mathrm{H}_{d-1}(Q[X_1, X_2, \dots, X_{d-1}])$ are finitely generated. More precisely, here is what we do at each stage:

Step d: Given sets of variables $X_1, \dots, X_{d-1}$ such that

$$\mathrm{H}_0(Q[X_1, X_2, \dots, X_{d-1}]) = R \quad \text{and} \quad \mathrm{H}_i(Q[X_1, X_2, \dots, X_{d-1}]) = 0 \text{ for all } i < d-1,$$

we fix cycles $u_1, \dots, u_t$ of degree $d-1$ in $Q[X_1, X_2, \dots, X_{d-1}]$ whose classes in homology generate $\mathrm{H}_{d-1}(Q[X_1, X_2, \dots, X_{d-1}])$, and add new variables $v_1, \dots, v_t$ of degree d to kill the homology in degree $d-1$:

$$\partial(v_i) = u_i.$$

We set $X_d = \{v_1, \dots, v_t\}$ and proceed with $Q[X_1, X_2, \dots, X_d] = Q[X_{\leqslant d}]$.

Our new variables satisfy only the relations they must:

- When d is odd, we take all v_i to be exterior variables.
- When d is even, we take all v_i to be polynomial variables or divided power variables, which we will describe below; typically we choose one or the other for *all* even degrees at once.

Finally, we set

$$X := \bigcup_{i \geqslant 1} X_i.$$

The resulting complex $Q[X]$ is a free resolution for R over Q with a dg algebra structure.

Remark 1.3.4. For each fixed i, once we are past the stage where we have added the variables in X_{i+1} to correct homology in degree i, we never again change the homology in degrees i and below. Thus

$$\mathrm{H}_i(Q[X]) = \mathrm{H}_i(Q[X_{\leqslant i+1}]) = \begin{cases} R & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

This shows that $Q[X]$ is indeed a free resolution for R over Q, and in fact that every cyclic module over a noetherian local ring has a dg algebra resolution.

However, note that the process might never terminate: we might need to add variables in infinitely many degrees. In fact, as we will see below, we often add variables in *every* degree.

Remark 1.3.5. When I is generated by a regular sequence, we may stop at step 1, since the Koszul complex is a resolution for $R = Q/I$. On the other hand, if the minimal generators for I do not form a regular sequence, by Theorem 1.1.4 the Koszul complex is not exact in degree 1, and thus we must add variables of degree 2.

Exercise 1.3.6. Let x be an element of even degree in a dg algebra with differential ∂. Show that

$$\partial(x^n) = n\partial(x)x^{n-1} \quad \text{for all } n \geqslant 1.$$

Remark 1.3.7 (Divided power variables)**.** The potential disadvantage of polynomial variables is only visible in prime characteristic. Each time we add a new variable x of even degree, its ripple effect is felt forever, as all the powers x^n are nonzero. This is sometimes an advantage: by the time we get to fixing the homology in some degree $d-1$, we might already have elements of degree d, made out of products of variables of smaller degrees, that turn those cycles into boundaries. But in prime characteristic p, the lower degree variables might have created new cycles as well: if x has even degree, then by Exercise 1.3.6

$$\partial(x^p) = p\partial(x)x^{p-1} = 0.$$

To avoid this, when in prime characteristic, rather than adding one variable x in even degree, we add an infinite collection of variables $x = x^{(1)}$ and $x^{(i)}$ for all $i \geqslant 1$, satisfying the following rules:

$$x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)} \qquad \text{and} \qquad \partial(x^{(i+1)}) = x^{(i)}\partial(x).$$

We say that $x^{(i)}$ are **divided power variables**. Note, however, that over a field of characteristic 0, this recipe coincides with adding polynomial variables, as

$$x^{(i)} = \frac{1}{i!}x^i.$$

We will write $S\langle x\rangle$ for the dg S-algebra obtained by adjoining the divided power variables $x^{(i)}$ to S, to distinguish from $S[x]$, obtained by adjoining the polynomial variable x.

Definition 1.3.8. Given a dg algebra A and a cycle $z \in \mathrm{Z}(A)$, a **simple semifree extension** of A is a dg algebra $B = A[x \mid \partial(x) = z]$ obtained by adjoining a variable x to kill the cycle z, where x is taken to be an exterior variable if $|x| = |z| + 1$ is odd and a polynomial variable if $|x| = |z| + 1$ is even. We may instead adjoin divided power variables in even degrees, in which case we will write $B = A\langle x \mid \partial(x) = z\rangle$; we will abuse notation and call both situations a simple semifree extension of A. Any dg algebra B obtained from A by a sequence of simple semifree extensions of A is a **semifree extension** of A. If X is the set of variables we adjoined to form B, we write $B = A[X]$ or $B = A\langle X\rangle$.

Remark 1.3.9. Note that the resolution $Q[X]$ for $R = Q/I$ over Q in Construction 1.3.3 is a semifree extension $Q[X]$ of Q with

$$\mathrm{H}(Q[X]) = R.$$

Notation 1.3.10. Given a semifree extension $A[X]$ of A, we will write X_i to denote the variables of homological degree i in X, and

$$X_{\geqslant d} = \bigcup_{i \geqslant d} X_i, \qquad X_{\leqslant d} = \bigcup_{i \leqslant d} X_i.$$

Moreover,

$$X_{i_1} \cdots X_{i_s} := \{x_{i_1} \cdots x_{i_s} \mid x_{i_j} \in X_j\}$$

and $AX_{i_1} \cdots X_{i_s}$ denotes the free graded A-module on the set $X_{i_1} \cdots X_{i_s}$. Analogously, we may write X^d or AX^d.

There are two instances of this construction that will be of special interest to us.

Definition 1.3.11. Let $\widehat{R} \cong Q/I$ be a minimal regular presentation for a local ring R.

- A **minimal model** $Q[X]$ for R over Q is a semifree extension over Q that resolves Q/I over Q, where we adjoin exterior variables in odd degrees and polynomial variables in even degrees, and take the smallest number possible of variables in each degree.
- An **acyclic closure** $R\langle Y\rangle$ for k over R is a semifree extension over R that resolves k over R, where we adjoin exterior variables in odd degrees and divided power variables in even degrees, and take the smallest number possible of variables in each degree.

Macaulay2. The Macaulay2 package `DGAlgebras` allows one to compute minimal modules.

```
o1 = DGAlgebras

o1 : Package

i2 : Q = QQ[x,y];

i3 : I = ideal"x2,xy";

o3 : Ideal of Q

i4 : A = minimalModel(I, EndDegree => 3);
```

The `EndDegree => 3` instruction tells Macaulay to compute $Q[X_{\leqslant 4}]$, so that we get zero homology up to degree 3. Macaulay2 creates a ring whose variables are indexed as $T_{i,j}$, with i indicating homological degree.

We can extract the differential of $R\langle Y_{\leqslant 4}\rangle$ via

```
i5 : d = A.diff;
```

and then apply it to a particular variable to find its image:

```
i6 : d(T_(1,2))

o6 = x*y

o6 : Q[T   ..T   , T   , T   , T   ..T   ]
        1,1   1,2   2,1   3,1   4,1   4,2
```

Warning: we cannot use `d` calculate the differential of a product of variables directly, due to the way that is coded into Macaulay2. However, we can compute it by calculating the image of each variable individually, and using the Leibniz rule.

Exercise 1.3.12. Let $Q = k[\![x,y]\!]$, $I = (x^2, xy)$, and $R = Q/I$.

1) Write the first 3 steps to construct a minimal model for R over Q.

2) Write the first 3 steps to construct an acyclic closure for k over R.

3) Check your work with the `DGAlgebras` package.

Example 1.3.13. Let $Q = k[\![x,y,z]\!]$ and $I = (xy, zx, yz)$. Let us compute the first few steps in a minimal model for $R = Q/I$ over Q. Set

$$f_1 = xy \qquad f_2 = xz \qquad f_3 = yz.$$

First, we take $X_1 = \{T_{1,1}, T_{1,2}, T_{1,3}\}$ with

$$\partial(T_{1,1}) = xy \qquad \partial(T_{1,2}) = xz \qquad \partial(T_{1,3}) = yz.$$

The boundaries of degree 1 in $Q[X_1]$ correspond to the koszul relations on R, which are generated by $f_i f_j - f_j f_i$ with $i \neq j$. The nonkoszul relations on f_1, f_2, f_3 are generated by

$$zf_1 - yf_2 \qquad yf_2 - xf_3 \qquad zf_1 - xf_3.$$

In fact, note that any two of these generate the module of relations on I. More precisely, $zT_1 - yT_2$ and $zT_1 - zT_3$ generate $\mathrm{H}_1(Q[X_1])$. Thus we take $X_2 = \{T_{2,1}, T_{2,2}\}$ with

$$\partial(T_{2,1}) = zT_{1,1} - yT_{1,2} \qquad \text{and} \qquad \partial(T_{2,2}) = zT_{1,1} - xT_{1,3}.$$

Now while we can compute $\mathrm{H}_2(Q[X_{\leqslant 2}])$ by hand, we can also get a little help from Macaulay2. We can directly set up the matrices involve, or set up the complex by hand ourselves, or use the `DGAlgebras` package to build it. Let us use the `DGAlgebras` package. First, we set up $Q[X_{\leqslant 2}]$:

```
i1 : needsPackage "DGAlgebras";

i2 : Q = QQ[x,y,z];

i3 : I = ideal"xy,xz,yz";

i4 : R = Q/I;

i5 : A = minimalModel(I, EndDegree => 1)

o5 = {Ring => Q                                                     }
      Underlying algebra => Q[T   ..T   , T   ..T   ]
                               1,1   1,3   2,1   2,2
      Differential => {x*y, x*z, y*z, - z*T    + y*T   , - z*T    + x*T   }
                                            1,1      1,2        1,1      1,3

o5 : DGAlgebra
```

Let us check which the choices that Macaulay2 has made:

```
i6 : d = A.diff;

o6 : RingMap Q[T   ..T   , T   ..T   ] <-- Q[T   ..T   , T   ..T   ]
                1,1   1,3   2,1   2,2         1,1   1,3   2,1   2,2

i7 : d(T_(2,1))

o7 = - z*T    + y*T
          1,1      1,2

o7 : Q[T   ..T   , T   ..T   ]
        1,1   1,3   2,1   2,2

i8 : d(T_(2,2))

o8 = - z*T    + x*T
          1,1      1,3

o8 : Q[T   ..T   , T   ..T   ]
        1,1   1,3   2,1   2,2
```

We can see that Macaulay2 has made the same choice we made, though with opposite signs. Before we compute $\mathrm{H}_2(Q[X_{\leqslant 2}])$, we should also check what order Macaulay2 is using for the elements in $QX_1^2 \oplus QX_2$. For example, we can see this by turning $A = Q[X]$ into the form of a complex, and asking for the differential ∂_2, as follows:

```
i9 : C = toComplex (A, 3)

      1      3      5      7
o9 = Q  <-- Q  <-- Q  <-- Q

     0      1      2      3

o9 : Complex

i10 : C.dd_2

o10 = {2} | -xz -yz 0   -z -z |
      {2} | xy  0   -yz y  0  |
      {2} | 0   xy  xz  0  x  |

               3      5
o10 : Matrix Q  <-- Q
```

We see that the basis in degree 2 has been chosen in the following order:

$$\{T_{1,1}T_{1,2}, \quad T_{1,1}T_{1,3}, \quad T_{1,2}T_{1,3}, \quad T_{2,1}, \quad T_{2,2}\}.$$

Now we are ready to compute $\mathrm{H}_2(Q[X_{\leqslant 2}])$:

```
i9 : mingens HH_2(A)

o9 = {4} | -1 0  0  |
     {4} | 0  0  -1 |
     {4} | 0  -1 0  |
     {3} | x  -z 0  |
     {3} | 0  z  y  |

              5      3
o9 : Matrix Q  <-- Q
```

We now set $X_3 = \{T_{3,1}, T_{3,2}, T_{3,3}\}$ with

$$\partial(T_{3,1}) = T_{1,1}T_{1,2} - xT_{2,1} \qquad \partial(T_{3,2}) = T_{1,2}T_{1,3} + zT_{2,1} - zT_{2,2} \qquad \partial(T_{3,2}) = T_{1,1}T_{1,3} - yT_{2,2}.$$

We can check by hand that these three elements are in fact cycles but not boundaries in $Q[X_{\leqslant 2}]$; the trickier part would have been to see that they do form a minimal generating set for the second homology.

A minimal model is typically not a minimal free resolution. However, analogously to what happens with minimal free resolutions, we can characterize minimal models according to the shape of the differential.

Exercise 1.3.14. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and fix a minimal regular presentation $\widehat{R} \cong Q/I$ for R. Let $(Q[X], \partial)$ be a minimal model for R over Q.

1) Show that $\partial(X_1) \subseteq \mathfrak{m}^2$ and $\partial(X_2) \subseteq \mathfrak{m}X_1$.

2) Give an example showing that $\partial(X_{n+1}) \subseteq \mathfrak{m}X_n$ might fail in general.

Remark 1.3.15. Given a minimal model $A := Q[X]$ for R over the regular local ring $(Q, \mathfrak{m}, k)$, we will use X_0 to denote a minimal set of generators for $\mathfrak{m}$. Note that $Q[X]$ is a local dg algebra, and this notation allows us to write the maximal ideal of $Q[X]$ as (X), where we now view $X = X_{\geqslant 0}$.

Similarly, $k[X] = Q[X] \otimes_Q k$ is also a local dg algebra, with maximal ideal $(X) = (X_{\geqslant 1})$.

Definition 1.3.16. Let $(R, \mathfrak{m})$ be a noetherian local ring, and let A be a dg R-algebra with $A_0 = R$. We say that A is a **minimal dg algebra** if the maximal ideal $J = \mathfrak{m} + (A_{\geqslant 1})$ satisfies

$$\partial(J) \subseteq J^2.$$

Note that this minimality condition on a dg algebra is far from requiring that it be a minimal complex. Recall that in general, a complex of modules over a local ring $(R, \mathfrak{m})$ is minimal if

$$\partial(F) \subseteq \mathfrak{m}F.$$

Remark 1.3.17. In a picture, here is what a minimal dg algebra $k[X]$ looks like, where we indicate with arrows the only components that might be nonzero.

$$\begin{tikzcd}[row sep=small]
 & kX_4 \arrow[ddddr] \arrow[ddddddr] & & & & \\
 & \oplus & & & & \\
 & kX_1X_3 \arrow[ddddr] & kX_3 \arrow[ddr] & kX_2 & & \\
 & \oplus & \oplus & \oplus & & \\
\cdots & kX_1^2X_2 & kX_1X_2 & kX_1^2 & kX_1 & R. \\
 & \oplus & \oplus & & & \\
 & kX_1^4 & kX_1^3 & & & \\
 & \oplus & & & & \\
 & kX_2^2 & & & &
\end{tikzcd}$$

Theorem 1.3.18. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $\widehat{R} \cong Q/I$ a minimal regular presentation. Let $Q[X]$ be the minimal model for R over Q, with maximal ideal $J = (X_{\geqslant 0})$. Then*

$$\partial_{Q[X]}(J) \subseteq J^2.$$

As a consequence, in $k[X] = Q[X] \otimes_Q k$, the maximal ideal $(X_{\geqslant 1})$ satisfies

$$\partial_{k[X]}(X) \subseteq (X)^2.$$

Proof. First, note that $\partial_{k[X]}(X_1) = 0$ by Exercise 1.3.14. Fix $d \geqslant 1$. For each $y \in X_{d+1}$, write $\partial(y) \in Q[X]$ as

$$\partial(y) = w + \sum_{x \in X_d} a_x x$$

for some $a_x \in Q$ and $w \in (X)^2$. Applying ∂ again, we get

$$0 = \partial(w) + \sum_{x \in X_d} a_x \partial(x) \in \mathrm{Z}_d\left(Q[X_{<d}]\right).$$

Note that by degree reasons, $w \in Q[X_{<d}]$, so $\partial(w)$ is a boundary in $Q[X_{<d}]$. Therefore,

$$0 = \sum_{x \in X_d} a_x [\partial(x)] \in \mathrm{H}_d\left(Q[X_{<d}]\right).$$

By assumption, $Q[X]$ is a minimal model for R, and thus $\{[\partial(x)] \mid x \in X_d\}$ forms a minimal generating set for $\mathrm{H}_d\left(Q[X_{<d}]\right)$. By NAK, $a_x \in \mathfrak{m}$ for all $x \in X_d$, and therefore

$$\partial(y) = w + \sum_{x \in X_d} a_x x \in (X)^2 + \mathfrak{m}X. \qquad \square$$

In fact, this condition characterizes minimal models.

Exercise 1.3.19. Let $Q[X]$ be a semifree extension of Q. Show that $Q[X]$ is a minimal model for $R = Q/I$ over Q if and only if

$$\mathrm{H}_n(Q[X]) = \begin{cases} R & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\partial(X) \subseteq (X)^2.$$

Our next goal is to show that minimal models are unique up to isomorphism. To build towwards that, we need to better understand the construction. As we briefly discussed earlier, the semifree extensions are free strictly graded commutative algebras, and thus they come equipped with the following universal property:

Theorem 1.3.20. *Let $\varphi\colon A \longrightarrow B$ be a map of dg algebras and let Z be a set of cycles in A whose image $\varphi(Z)$ is a set of boundaries in B. For each $z \in Z$, let $b_z \in B$ be such that $\partial(b_z) = \varphi(z)$. Given a set of variables indexed by Z*

$$X = \{x_z \mid z \in Z, |x_z| = |z| + 1\}$$

there exists a unique map of dg algebras $\widetilde{\varphi}\colon A[X \mid \partial(x_z) = z] \longrightarrow B$ extending φ and such that

$$\widetilde{\varphi}(x_z) = b_z.$$

Proof. We can construct the extension inductively, one variable at a time, so it is sufficient to give a proof in the case of one variable. Let $z \in \mathrm{Z}(A)$ be such that $\varphi(z) = \partial(b)$ for some $b \in B$. Note that $A[x \mid \partial(x) = z]$ is a free graded commutative A-algebra, so φ extends uniquely to a map of graded commutative algebras

$$\widetilde{\varphi}\colon A[x \mid \partial(x) = z] \longrightarrow B$$

with $\widetilde{\varphi}(x) = b$. Moreover, we claim that $\widetilde{\varphi}(x)$ is a map of complexes. By linearity, it is sufficient to consider elements of the form ax^i for some $i \geqslant 1$. We have

$$\begin{aligned}
\partial \circ \widetilde{\varphi}(ax^i) &= \partial(\varphi(a)b^i) \\
&= \partial(\varphi(a))b^i + (-1)^{|a|}\varphi(a)\partial(b^i) && \text{by the Leibniz rule} \\
&= \varphi(\partial(a))b^i + (-1)^{|a|}\varphi(a)\partial(b^i) && \text{since } \varphi\partial = \partial\varphi \\
&= \varphi(\partial(a))b^i + (-1)^{|a|}i\varphi(a)\partial(b)b^{i-1} && \text{by Exercise 1.3.6} \\
&= \varphi(\partial(a))\varphi(x^i) + (-1)^{|a|}i\varphi(a)\varphi(\partial(x))\widetilde{\varphi}(x^{i-1}) && \text{since } \varphi(\partial(x)) = \varphi(z) = \partial(b) \\
&= \widetilde{\varphi}\left(\partial(a)x^i + (-1)^{|a|}iaz\varphi(\partial(x))x^{i-1}\right) \\
&= \widetilde{\varphi}\left(\partial(ax^i)\right) && \text{by Exercise 1.3.6.}
\end{aligned}$$

We conclude that $\widetilde{\varphi}$ is indeed a map of dg algebras. ☐

In colloquial terms, Theorem 1.3.20 tells us that to give a map of dg algebras $A[X] \longrightarrow B$ is to give a map of dg algebras $A \longrightarrow B$ together with the images of each $x \in X$, such that the image of each $x \in X$ are that $\partial(x)$ needs to be sent to a boundary with $\varphi(\partial(x)) = \partial(\varphi(x))$.

Remark 1.3.21. Let A be a dg algebra and consider a semifree extension

$$C = A[x \mid \partial(x) = y].$$

When $|x|$ is odd, $C = A \oplus Ax$, and the inclusion of A into C induces a natural short exact sequence of complexes

$$0 \longrightarrow A \longrightarrow A[x] \longrightarrow \Sigma^{|x|} A \longrightarrow 0.$$

More generally, for any x, the truncations

$$\bigoplus_{i \leqslant n} Ax^i$$

obtained by quotenting $A[x]$ by (x^{n+1}) give us short exact sequences of dg A-modules

$$0 \longrightarrow \bigoplus_{i<n} Ax^i \longrightarrow \bigoplus_{i \leqslant n} Ax^i \longrightarrow Ax^n \longrightarrow 0.$$

Note that when $n = 1$, this is the same as the first short exact sequence above for $|x|$ odd. Note moreover that

$$A[x] = \bigoplus_{i} Ax^i$$

and that whenever $\ell \leqslant n-1$,

$$\mathrm{H}_\ell(A[x]) = \mathrm{H}_\ell\left(\bigoplus_{i \leqslant n} Ax^i\right).$$

Corollary 1.3.22. *Let $\alpha\colon A \longrightarrow B$ be a map of dg algebras and let Z be a set of cycles in A. Then given a set of variables indexed by Z*

$$X = \{x_z \mid z \in Z\}$$

there exists a unique map of dg algebras

$$\begin{aligned} A[X \mid \partial(x_z) = z] &\xrightarrow{\widetilde{\alpha}} B[X \mid \partial(x_z) = \alpha(z)] \\ x_z &\longmapsto x_z \end{aligned}$$

extending α. Moreover, if α is surjective, a quasiisomorphism, or a local map, then so is $\tilde{\alpha}$.

Proof. Consider the composition of α with the natural inclusion of B into $B[X]$:

$$\beta\colon A \xrightarrow{\alpha} B \hookrightarrow B[X].$$

For each $z \in Z$, since z is a cycle and α is a map of dg algebras, $\alpha(z)$ is also a cycle, which becomes a boundary in $B[X]$. By Theorem 1.3.20, there exists a unique map of dg algebras $\gamma\colon A[X] \longrightarrow B[X]$ extending β and such that $\gamma(x_z) = x_z$ for each $z \in Z$. By construction, γ extends α. Note moreover that any map of dg algebras $A[X] \longrightarrow B[X]$ extending α must also extend β, and thus γ is the unique extension of α.

Now assume that α is a quasiisomorphism. To show that $\gamma = \tilde{\alpha}$ is also a quasiisomorphism, it suffices to consider the case of one variable x. Consider the restrictions of $\tilde{\alpha}$ given by

$$\bigoplus_{i \leqslant n} Ax^i \xrightarrow{\alpha_n} \bigoplus_{i \leqslant n} Bx^i.$$

By Remark 1.3.21, we have following commutative diagram of short exact sequences of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{i<n} Ax^i \arrow[r] \arrow[d, "\alpha_{n-1}"] & \bigoplus\limits_{i\leqslant n} Ax^i \arrow[r] \arrow[d, "\alpha_n"] & Ax^n \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \bigoplus\limits_{i<n} Bx^i \arrow[r] & \bigoplus\limits_{i\leqslant n} Bx^i \arrow[r] & Bx^n \arrow[r] & 0.
\end{tikzcd}$$

Note that the map on the right is a quasiisomorphism since it is given by a shift of α. Following an argument by induction, we may assume that α_n is a quasiisomorphism; note that $\alpha_0 = \alpha$ is a quasiisomorphism by assumption. This gives rise to the following diagram of long exact sequences in homology:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{\ell-1}(A) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_\ell\left(\bigoplus\limits_{i<n} Ax^i\right) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_\ell\left(\bigoplus\limits_{i\leqslant n} Ax^i\right) \arrow[r] \arrow[d, "\mathrm{H}(\alpha_n)"] & \mathrm{H}_\ell(A) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_{\ell+1}\left(\bigoplus\limits_{i<n} Ax^i\right) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_{\ell-1}(B) \arrow[r] & \mathrm{H}_\ell\left(\bigoplus\limits_{i<n} Bx^i\right) \arrow[r] & \mathrm{H}_\ell\left(\bigoplus\limits_{i\leqslant n} Bx^i\right) \arrow[r] & \mathrm{H}_\ell(B) \arrow[r] & \mathrm{H}_{\ell+1}\left(\bigoplus\limits_{i<n} Bx^i\right) \arrow[r] & \cdots.
\end{tikzcd}$$

By the Five Lemma, $\mathrm{H}_\ell(\alpha_n)$ must be an isomorphism. Note that for all $\ell \leqslant n$,

$$\mathrm{H}_\ell(\alpha_n) = \mathrm{H}_\ell(\alpha).$$

Proceeding by induction, we conclude that $\tilde{\alpha}$ is a quasiisomorphism.

Finally, note that $B[X]$ is generated by B and X, and that the image of $\tilde{\alpha}$ is generated by $\alpha(A)$ and (X). Therefore, if α is surjective then so is $\tilde{\alpha}$, and if α is local then so is $\hat{\alpha}$. ☐

Exercise 1.3.23. Let $\alpha\colon A \longrightarrow B$ be a map of dg algebras and let Z be a set of cycles in A. Show that given given a set of variables X indexed by Z, there exists a unique map of dg algebras

$$\begin{aligned} A\langle X \mid \partial(x_z) = z\rangle &\xrightarrow{\tilde{\alpha}} B\langle X \mid \partial(x_z) = \alpha(z)\rangle \\ x_z &\longmapsto x_z \end{aligned}$$

extending α. Show that if α is surjective, local, or a quasiisomorphism, then so is $\tilde{\alpha}$.

Exercise 1.3.24. Let F be a complex of free R-modules with $F_n = 0$ for all $n < 0$ and let $f\colon A \longrightarrow B$ be a quasiisomorphism of R-complexes.

1) Consider the truncation $F_{\leqslant n}$ of F that has F_d in homological degrees $d \leqslant n$, and zeroes elsewhere. Show that the natural inclusion of $F_{\leqslant n}$ into $F_{\leqslant n+1}$ induces a split short exact sequence of complexes.

2) Show that
$$f \otimes_R \mathrm{id}_F\colon A \otimes_R F \longrightarrow B \otimes_R F$$
is a quasiisomorphism.

3) Show that
$$f_* := \mathrm{Hom}_R(F, f)\colon \mathrm{Hom}_R(F, A) \longrightarrow \mathrm{Hom}_R(F, B)$$
is a quasiisomorphism.

Exercise 1.3.25. Let $\pi : A \longrightarrow B$ be a map of complexes such that $\pi_* = \mathrm{Hom}(C, \pi)$ is a quasiisomorphism. Show that if $\alpha \in \ker(\pi_*)$ then α is nullhomotopic.

Theorem 1.3.26. *Let $(Q, \mathfrak{m}, k)$ be a commutative ring and $R = Q/I$. Let $Q[X]$ be a semifree extension of Q that is a free resolution for R over Q, and let $\pi : A \twoheadrightarrow B$ be a surjection quasiisomorphism of dg algebras. Every dg algebra map $\beta\colon Q[X] \longrightarrow B$ lifts to a dg algebra map $\alpha\colon Q[X] \longrightarrow A$ making the following diagram commute:*

$$\begin{tikzcd} & A \arrow[d, two heads, "\pi"] \\ Q[X] \arrow[ur, dashed, "\alpha"] \arrow[r, "\beta"'] & B \end{tikzcd}$$

Moreover, any such lift α is unique up to Q-linear homotopy.

Proof. Since Q is a free Q-module, there exists map of Q-complexes making

$$\begin{tikzcd} & A_0 \arrow[d, "\pi_0"] \\ Q \arrow[ur, dashed, "\alpha_0"] \arrow[r, "\beta_0"'] & B_0 \end{tikzcd}$$

commute. Note that α_0 determines a map of dg algebras $\alpha_0\colon Q \longrightarrow A$.

Now we construct $\alpha_n\colon Q[X_{\leqslant n}] \longrightarrow A$ inductively. Suppose that α_n has been set, and fix a variable $x \in X_{n+1}$. Since π is surjective, there exists an element $w_x \in A$ such that $\pi(w_x) = \beta(x)$. Note that
$$\alpha_n(\partial(x)) = \partial(w_x)$$
is a boundary. By Theorem 1.3.20, there exists a unique map of dg algebras
$$\alpha_{n+1}\colon Q[X_{\leqslant n+1}] \longrightarrow A$$
extending α_n and such that $\alpha(x) = z_x$. Note that α_n extends β by construction. Proceeding inductively leads us to $\alpha\colon Q[X] \longrightarrow A$.

Now suppose that $\gamma\colon Q[X] \longrightarrow A$ is another lift of β. Since $\pi \circ (\alpha - \gamma) = 0$, $\alpha - \gamma \in \ker(\pi_*)$. By Exercise 1.3.24, π_* is a quasiisomorphism, so $\alpha - \gamma$ is nullhomotopic by Exercise 1.3.25. Thus α and γ are homotopic. □

Notation 1.3.27. Given two maps of complexes $f, g\colon A \longrightarrow B$, we write $f \sim g$ to denote that f and g are homotopic.

Definition 1.3.28. A map of complexes $\alpha\colon A \longrightarrow B$ is a **homotopy equivalence** if there exists a map of complexes $\beta\colon B \longrightarrow A$ such that $\alpha \circ \beta \sim \mathrm{id}_B$ and $\beta \circ \alpha \sim \mathrm{id}_A$. We say A and B are **homotopy equivalent** if there exists a homotopy equivalence between them. Whenever A and B are dg algebras, a **homotopy equivalence of dg algebras** is a homotopy equivalence such that α and β as above are maps of dg algebras.

Exercise 1.3.29. Let A be a dg algebra. Show that the natural map

$$A[x]\langle y \mid \partial(y) = x\rangle \longrightarrow A$$

is a homotopy equivalence of dg algebras. Hint: consider the cases when $|x|$ is even and odd separately.

Remark 1.3.30. Any homotopy equivalence is a quasiisomorphism, since homology is a functor. Note that any two semifree extensions of Q resolving $R = Q/I$ over Q are quasiisomorphic: by Theorem 1.3.26, there are maps of dg algebras α and β making the following diagrams commute:

$$\begin{tikzcd} & Q[Y] \arrow[d, two heads, "\simeq"] \\ Q[X] \arrow[ur, dashed, "\alpha"] \arrow[r, "\simeq"'] & R \end{tikzcd} \qquad \begin{tikzcd} & Q[X] \arrow[d, two heads, "\simeq"] \\ Q[Y] \arrow[ur, dashed, "\beta"] \arrow[r, "\simeq"'] & R. \end{tikzcd}$$

By Theorem 1.3.26, α and β are quasiisomorphisms since they are lifts of quasiisomorphisms.

Lemma 1.3.31. *Any two semifree extensions A and B of Q resolving $R = Q/I$ over Q are homotopy equivalent. Moreover, any quasiisomorphism $A \longrightarrow B$ is a homotopy equivalence.*

Proof. We saw in Remark 1.3.30 that there exist quasiisomorphisms $\alpha\colon A \longrightarrow B$ and $\beta\colon B \longrightarrow A$, so it suffices to prove the second statement.

When we compose α and β, we get commutative diagrams

$$\begin{tikzcd} & Q[X] \arrow[d, two heads, "\simeq"] \\ Q[X] \arrow[ur, dashed, "\beta\circ\alpha"] \arrow[r, "\simeq"'] & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Q[Y] \arrow[d, two heads, "\simeq"] \\ Q[Y] \arrow[ur, dashed, "\alpha\circ\beta"] \arrow[r, "\simeq"'] & R. \end{tikzcd}$$

Note that the identity on $Q[X]$ or $Q[Y]$ are also maps that makes each of these diagrams commute, so by the uniqueness in Theorem 1.3.26 we conclude that $\beta \circ \alpha$ is homotopic to $\mathrm{id}_{Q[X]}$ and $\alpha \circ \beta$ is homotopic to $\mathrm{id}_{Q[Y]}$. □

Exercise 1.3.32. Let A be a dg algebra and assume $\alpha\colon M \to N$ is a homotopy equivalence of dg A-modules. Prove that for any dg ideal J of A the induced map $\overline{\alpha}\colon M/JM \to N/JN$ is a homotopy equivalence of dg A/J-modules.

Theorem 1.3.33. *Let R be a noetherian local ring, and $\widehat{R} \cong Q/I$ a minimal regular presentation. Any two minimal models for R over Q are isomorphic dg algebras.*

Proof. Let $Q[X]$ and $Q[Y]$ be two minimal models for R over Q. We claim that it suffices to show that any homotopy equivalence $Q[X] \longrightarrow Q[X]$ is an isomorphism. Indeed, by Lemma 1.3.31, $Q[X]$ and $Q[Y]$ are homotopy equivalent, so given homotopy equivalences $\alpha\colon Q[X] \longrightarrow Q[Y]$ and $\beta\colon Q[X] \longrightarrow Q[Y]$, we will have shown that $\alpha \circ \beta$ and $\beta \circ \alpha$ are isomorphisms, which implies that α and β are isomorphisms.

Let $\varphi\colon Q[X] \longrightarrow Q[X]$ be a homotopy equivalence. We will work in $k[X] = Q[X] \otimes_Q k$. By Theorem 1.3.18, $J = (X)$ is a dg ideal in $k[X]$. Consider the inverse limit

$$k[X]/J \longleftarrow k[X]/J^2 \longleftarrow k[X]/J^3 \longleftarrow \cdots$$

We claim that[1]

$$\varprojlim k[X]/J^n \cong k[X].$$

Indeed, this inverse limit can be calculated degreewise, so fix a degree d. Note that since J lives only in positive degrees, then for $n \gg 0$ we have $(J^n)_d = 0$, so $(k[X]/J^n)_d$ is eventually constant and equal to $k[X]_d$.

Consider the induced maps

$$\varphi_n\colon k[X]/J^n \longrightarrow k[X]/J^n$$

which are still homotopy equivalences by Exercise 1.3.32. We claim that all of these induced maps are isomorphisms. To see that, first consider $n = 1$, and note that the induced map is simply the identity on k, which is in fact an isomorphism.

Now we proceed by induction. Assume that φ_n is an isomorphism. Since φ respects the algebra structure, $\varphi(J^n) \subseteq J^n$ for all n, so φ_{n+1} induces a map $J^n/J^{n+1} \longrightarrow J^n/J^{n+1}$, which must also be a homotopy equivalence. Consider the commutative diagram short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & J^n/J^{n+1} \arrow[r] \arrow[d, "\varphi_{n+1}"] & k[X]/J^{n+1} \arrow[r] \arrow[d, "\varphi_{n+1}"] & k[X]/J^n \arrow[r] \arrow[d, "\varphi_n"] & 0 \\
0 \arrow[r] & J^n/J^{n+1} \arrow[r] & k[X]/J^{n+1} \arrow[r] & k[X]/J^n \arrow[r] & 0.
\end{tikzcd}$$

By the Five Lemma, it suffices to show that the restriction of φ_{n+1} to J^n/J^{n+1} is also an isomorphism.

[1] Warning: for those familiar with completions, this will be very disconcerting. The key point to keep in mind is that the grading plays an important role here.

By Theorem 1.3.18 $\partial(J) \subseteq J^2$, so it follows that J^n/J^{n+1} has trivial differential for all $n \geqslant 1$. Therefore, $\mathrm{H}(J^n/J^{n+1}) = J^n/J^{n+1}$, and thus the map induced in homology by

$$J^n/J^{n+1} \xrightarrow{\varphi_n} J^n/J^{n+1}$$

can be identified with itself. By assumption, φ_n is a homotopy equivalence, and thus φ_n is an isomorphism. Therefore, $\varphi \otimes_Q \mathrm{id}_k$ is an isomorphism. Applying NAK degreewise, we conclude that φ is an isomorphism. $\square$

Similarly, acyclic closures are also unique.

Exercise 1.3.34. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Show that any two acyclic closures $R\langle Y\rangle$ and $R\langle Z\rangle$ for k over R are isomorphic dg algebras. More precisely, there is an isomorphism of dg algebras $\alpha\colon R\langle Y\rangle \longrightarrow R\langle Z\rangle$ such that the induced map $\tilde{\alpha}\colon k\langle Y\rangle \longrightarrow k\langle Z\rangle$ is an isomorphism of graded k-vector spaces.

Lemma 1.3.35. *Let R be a noetherian local ring and let $\widehat{R} \cong Q/I$ be a minimal regular presentation. The following are equivalent:*

1) *The ideal I is generated by a regular sequence.*
2) *We have $X = X_1$, so $Q[X]$ is the Koszul complex on a minimal generating set for I.*
3) *The minimal model $Q[X]$ for R is a finite free resolution for Q/I over Q.*
4) *The minimal model $Q[X]$ for R is a minimal free resolution for Q/I over Q.*

Proof. Let $\underline{f} = f_1, \ldots, f_n \in I$ be such that $\partial(X_1) = \{f_1, \ldots, f_n\}$. Note that $\underline{f}$ is a minimal generating set for I and $Q[X_1] = \mathrm{Kos}(\underline{f})$. Then

$$\begin{aligned}
\mathrm{H}_1(Q[X_1]) = 0 &\iff \underline{f} \text{ is a regular sequence} && \text{by Theorem 1.1.4} \\
&\iff I \text{ is generated by a regular sequence} && \text{by Corollary 0.3.10} \\
&\iff \mathrm{H}_i(Q[X_1]) = 0 \text{ for all } i \geqslant 1 && \text{by Corollary 0.3.10.}
\end{aligned}$$

This shows that 1) $\Leftrightarrow$ 2), 1) $\Rightarrow$ 3), and 1) $\Rightarrow$ 4). Now we prove 4) $\Rightarrow$ 3) $\Rightarrow$ 2).

Suppose $Q[X]$ is a minimal free resolution for Q/I. By Theorem 0.2.20, since Q is a regular local ring, $Q[X]$ must be a finite free resolution. Then $X_2 = \emptyset$, since adjoining any polynomial variable of even degree results in an infinite resolution. Thus $\mathrm{H}_1(Q[X_1]) = 0$, so $Q[X_1]$ is a resolution for R/I, by Theorem 1.1.4, and therefore $X = X_1$. $\square$

Complete intersections are the rings satisfying the equivalent properties in Lemma 1.3.35. Throughout these lectures, we will gain an understanding of complete intersections and give many equivalent definitions. Throughout, a pattern will emerge: there is a dichotomy between complete intersections and all other noetherian local rings, where complete intersections are the nicely behaved rings, and once R is not a complete intersection, then its behavior with respect to many interesting properties becomes completely wild. But first, let us dive deeper into the definition of our new favorite class of rings.

1.4 Complete intersections

Among singular rings, complete intersections are the mildest singularities. One of our main goals is to make this precise, and to give several characterizations of this class of rings.

Definition 1.4.1. A noetherian local ring R is a **complete intersection** of **codimension** c if for some regular presentation $\widehat{R} \cong Q/I$ for R, the ideal I is generated by a regular sequence of length c.

It is not clear from the definition that this notion does not depend on the choice of minimal regular presentation; we will show soon that this is indeed the case. First, let us see some examples.

Example 1.4.2. Any regular local ring is a complete intersection: we will see below in Remark 1.4.13 that $\widehat{R}$ must also be complete, so in this case, our usual defining ideal I is simply $I = 0$.

Example 1.4.3. A **hypersurface** is a complete intersection of codimension 1, meaning there is some nonzero $f \in \mathfrak{m}_Q^2$ such that $\widehat{R} \cong Q/(f)$. Conversely, note that any (complete) regular local ring $(Q, \mathfrak{m})$ is a domain, and thus and any nonzero $f \in \mathfrak{m}^2$ is a regular element, so that $Q/(f)$ is a complete intersection.

Example 1.4.4. Let k be a field and take $R = Q/I$ with $Q = k[\![x, y]\!]$, and $I = (x^2, xy)$. Note that $I = (x^2, xy)$ has two minimal generators but height 1; in fact, $\operatorname{Min}(I) = \{(x)\}$. Therefore, $\operatorname{height}(I) = 1$ and I is not generated by a regular sequence. The ring $R = Q/I$ is not a complete intersection.

Exercise 1.4.5. Let k be any field. Show that

$$R = k[\![x, y, z]\!]/(x^2, y^2, z^2, xyz)$$

is not a complete intersection ring.

We can give a more intrinsic definition of complete intersections via Koszul homology.

Definition 1.4.6. Let R be a noetherian local ring, and let $\underline{x}$ be any minimal generating set for the maximal ideal of R. We set

$$K^R := \operatorname{Kos}^R(\underline{x}).$$

The **Koszul homology of** R is the graded k-algebra $\mathrm{H}(K^R)$.

Exercise 1.4.7. Show that $\mathrm{H}(K^R)$ is a finite dimensional graded commutative k-algebra with $\mathrm{H}_0(K^R) = k$.

Remark 1.4.8. Depth sensitivity is one of the many nice properties of the Koszul complex we did not have a chance to discuss. In this setting, it tells us that

$$\sup\{i \mid \mathrm{H}_i(K^R) \neq 0\} = \operatorname{embdim}(R) - \operatorname{depth}(R).$$

This number is often called the **codepth** of R, written $\operatorname{codepth}(R)$.

Exercise 1.4.9. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and let $\underline{x}$ and $\underline{y}$ be two minimal generating sets for the same ideal. Show that $\operatorname{Kos}(\underline{x})$ and $\operatorname{Kos}(\underline{y})$ are isomorphic dg algebras.

Remark 1.4.10. By Exercise 1.4.9, any two minimal generating sets for the maximal ideal of R lead to isomorphic Koszul complexes. In particular, K^R is only well-defined up to isomorphism of dg algebras.

Lemma 1.4.11. *A noetherian local ring $(R, \mathfrak{m}, k)$ is regular if and only if $\mathrm{H}_1(K^R) = 0$.*

Proof. Fix a minimal generating set for $\mathfrak{m}$, say $\underline{x}$. Then

$$\begin{aligned} R \text{ is regular} &\iff \underline{x} \text{ is a regular sequence} && \text{by Theorem 0.3.12} \\ &\iff \mathrm{H}_1(\operatorname{Kos}(\underline{x})) = 0 && \text{by Theorem 1.4.18} \\ &\iff \mathrm{H}_1(K^R) = 0 && \text{by definition.} \quad \square \end{aligned}$$

In what follows, we will use the following well-known properties of $\mathfrak{m}$-adic completion:

- The completion $\widehat{R}$ is a flat R-algebra.
- For any finitely generated R-module M, its completion $\widehat{M}$ satisfies $\widehat{M} \cong M \otimes_R \widehat{R}$.
- The completion of a noetherian local ring $(R, \mathfrak{m})$ is a noetherian local ring $(\widehat{R}, \widehat{\mathfrak{m}})$ with

$$\dim(\widehat{R}) = \dim(R) \qquad \operatorname{embdim}(\widehat{R}) = \operatorname{embdim}(R) \qquad \operatorname{depth}(\widehat{R}) = \operatorname{depth}(R).$$

- For all $n \geqslant 1$, we have $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \widehat{\mathfrak{m}}^n/\widehat{\mathfrak{m}}^{n+1}$.

For those new to the topic of completion, we recommend focusing on the case of complete noetherian local rings, and more concretely on the special case of those containing a field, which are of the form $R = k[\![x_1, \dots, x_e]\!]/I$ with $I \subseteq (x_1, \dots, x_e)^2$. Nevertheless, we include here some of the technical details that allow us to reduce to the complete case.

Remark 1.4.12. Let M be a finitely generated R-module. We can take a minimal free resolution of M over R and tensor it with the flat R-module $\widehat{R}$ to obtain a free resolution over $\widehat{M}$, which is still minimal. Thus

$$\beta_i^R(M) = \beta_i^{\widehat{R}}(\widehat{M})$$

for all i. In particular,

$$\mu_R(M) = \beta_0^R(M) = \beta_0^{\widehat{R}}(\widehat{M}) = \mu_{\widehat{R}}(\widehat{M}).$$

Remark 1.4.13. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. By Remark 1.4.12, $\mu(\widehat{\mathfrak{m}}) = \mu(\mathfrak{m})$, so $\operatorname{embdim}(\widehat{R}) = \operatorname{embdim}(R)$. Since $\dim(R) = \dim(\widehat{R})$, we conclude that R is regular if and only if $\widehat{R}$ is regular.

Theorem 1.4.14. *Let R be a noetherian local ring. Then there is a quasiisomorphism of dg algebras*

$$K^R \longrightarrow K^{\widehat{R}}.$$

In particular, for all i,

$$\mu_R(\mathrm{H}_i(K^R)) = \dim_k(\mathrm{H}_i(K^R))) = \dim_k(\mathrm{H}_i(K^{\widehat{R}}))) = \mu_{\widehat{R}}(\mathrm{H}_i(K^{\widehat{R}})).$$

Proof. Fix a minimal generating set $\underline{x}$ for the maximal ideal of R. Let $\underline{\widehat{x}}$ be the image of $\underline{x}$ in $\widehat{R}$, and consider the map of complexes obtained by tensoring K^R with the canonical inclusion $R \longrightarrow \widehat{R}$:

$$K^R \cong K^R \otimes_R R \longrightarrow K^R \otimes_R \widehat{R}.$$

Note that $\underline{\widehat{x}}$ is a minimal generating set for the maximal ideal $\widehat{m}$ of $\widehat{R}$, and there are natural isomorphisms

$$K^R \otimes_R \widehat{R} \cong \mathrm{Kos}^R(\underline{x}) \otimes_R \widehat{R} \cong \mathrm{Kos}^{\widehat{R}}(\underline{\widehat{x}}) \cong K^{\widehat{R}}.$$

Thus we get a map of complexes $K^R \longrightarrow K^{\widehat{R}}$ making the following diagram commute:

$$\begin{tikzcd}
K^R \otimes_R R \arrow[r] \arrow[dr] & K^R \otimes_R \widehat{R} \arrow[d] \\
 & K^{\widehat{R}}
\end{tikzcd}$$

We claim this map $K^R \longrightarrow K^{\widehat{R}}$ is a quasiisomorphism.

First, note that since $\widehat{R}$ is a flat R-module, tensoring with $\widehat{R}$ commutes with taking homology. In particular,

$$\mathrm{H}_i(K^R) \otimes_R \widehat{R} = \mathrm{H}_i(\mathrm{Kos}(\underline{x})) \otimes_R \widehat{R} = \mathrm{H}_i(\mathrm{Kos}(\underline{x}) \otimes_R \widehat{R}) = \mathrm{H}_i(K^{\widehat{R}}).$$

By flatness of $\widehat{R}$ over R, the induced map $\mathrm{H}(K^R) \otimes_R R \longrightarrow \mathrm{H}(K^R) \otimes_R \widehat{R}$ is an isomorphism. Thus we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}(K^R) \otimes_R \widehat{R} \arrow[d, "\cong"] \\
\mathrm{H}(K^R) \otimes_R R \arrow[ur, "\cong"] \arrow[r] \arrow[dr] & \mathrm{H}(K^R \otimes_R \widehat{R}) \arrow[d, "\cong"] \\
 & \mathrm{H}(K^{\widehat{R}})
\end{tikzcd}$$

and therefore the remaining maps must also be isomorphisms. We conclude that the canonical map $K^R \longrightarrow K^{\widehat{R}}$ is a quasiisomorphism and that the finite dimensional k-vector spaces $\mathrm{H}(K^R)$ and $\mathrm{H}(K^{\widehat{R}})$ are isomorphic. $\square$

More generally, Theorem 1.4.14 is a corollary of the following:

Lemma 1.4.15. *Let R be a noetherian local ring and M be a complex of R-modules. If $\mathrm{H}_n(M)$ is a finite length R-module for each n, then the natural map*

$$M \longrightarrow M \otimes_R \widehat{R}$$

is a quasiisomorphism.

Sketch of proof. We follow the same proof strategy of Theorem 1.4.18, but this time using the commutative diagram

$$\begin{tikzcd} & \mathrm{H}(M) \otimes_R \widehat{R} \arrow[d, "\cong"] \\ \mathrm{H}(M) \otimes_R R \arrow[ur, "\cong"] \arrow[r] & \mathrm{H}(M \otimes_R \widehat{R}). \end{tikzcd}$$ □

Theorem 1.4.16. *Let $(Q, \mathfrak{m}, k)$ a regular ring and $R = Q/I$ for some ideal $I \subseteq \mathfrak{m}^2$. Fix a minimal generating set $\underline{y}$ for $\mathfrak{m}$. There is an isomorphism of dg Q-algebras*

$$R \otimes_Q \mathrm{Kos}(\underline{y}; Q) \longrightarrow K^R.$$

In particular, for all i we have

$$\mathrm{Tor}_i^Q(R, k) = \mathrm{H}_i(K^R).$$

Proof. Let $\underline{x}$ be the image of $\underline{y}$ in R. The natural isomorphisms $Q^n \otimes_Q R \cong R^n$ induce an isomorphism of complexes

$$\mathrm{Kos}^Q(\underline{y}) \otimes_Q R \longrightarrow K^R = \mathrm{Kos}^R(\underline{x})$$

that preserves the product structure. Taking homology on both sides, we conclude that

$$\mathrm{Tor}_i^Q(R, k) = \mathrm{H}_i(K^R).$$ □

Exercise 1.4.17. Let $(Q, \mathfrak{m}, k)$ be a noetherian local ring and $R = Q/I$. Show that

$$\mathrm{Tor}_1^Q(R, k) \cong I/\mathfrak{m}I.$$

We can now show that the minimal number of defining equations for $\widehat{R}$ over any minimal regular presentation is an invariant of R.

Theorem 1.4.18. *Let R be a noetherian local ring and $\widehat{R} \cong Q/I$ a minimal regular presentation for R. There is natural identification*

$$\mathrm{H}_1(K^R) \cong I/\mathfrak{m}I.$$

In particular,

$$\mu(\mathrm{H}_1(K^R)) = \mu(I).$$

Proof. By Theorem 1.4.14, we can reduce to the case when R is complete, so we may assume that $R = Q/I$. By Theorem 1.4.16 and Exercise 1.4.17,

$$\mathrm{H}_1(K^R) = \mathrm{Tor}_1^Q(R, k) \cong I/\mathfrak{m}I. \qquad \square$$

Theorem 1.4.19. *A noetherian local ring R is a complete intersection if and only if*

$$\mu(\mathrm{H}_1(K^R)) = \mathrm{embdim}(R) - \dim(R).$$

Proof. Let $\widehat{R} \cong Q/I$ be a minimal regular presentation for I, meaning that $(Q, \mathfrak{m})$ is a regular local ring and $I \subseteq \mathfrak{m}^2$. By Exercise 0.2.19, $\dim(Q) = \mathrm{embdim}(Q) = \mathrm{embdim}(R)$.

Since Q is a regular ring and $\dim(R) = \dim(\widehat{R})$, the dimension formula from Theorem 0.3.9 gives us

$$\mathrm{height}(I) = \dim(Q) - \dim(Q/I) = \mathrm{embdim}(R) - \dim(R).$$

On the other hand, by Corollary 0.3.10

$$I \text{ is generated by a regular sequence} \iff \mu(I) = \mathrm{height}(I).$$

By Theorem 1.4.18, $\mu(I) = \mu(\mathrm{H}_1(K^R))$. Thus

$$I \text{ is generated by a regular sequence} \iff \mu(\mathrm{H}_1(K^R)) = \mathrm{embdim}(R) - \dim(R). \qquad \square$$

Corollary 1.4.20. *Let R be a noetherian local ring and consider two regular local rings $(Q, \mathfrak{m})$ and $(S, \mathfrak{n})$ such that*

$$Q/I \cong \widehat{R} \cong S/J.$$

Then I is generated by a regular sequence if and only if J is generated by a regular sequence.

Proof. First, we claim that we can reduce to the case when both regular presentations are minimal. Indeed, note that if $I = L + (f)$ for some $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, then f is a regular element in Q. Note moreover that such an f is necessarily a minimal generated of I. Thus I is generated by a regular sequence if and only if L is generated by a regular sequence. Finally, $Q/(f)$ is a regular local ring by Exercise 0.2.17.

By Exercise 0.2.19, our assumptions on I and J guarantee that

$$\dim(Q) = \mathrm{embdim}(Q) = \mathrm{embdim}(\widehat{R}) = \mathrm{embdim}(S) = \dim(S).$$

Since Q and S are both regular rings, the dimension formula from Theorem 0.3.9 gives us

$$\operatorname{height}(I) = \dim(Q) - \dim(R) = \dim(S) - \dim(R) = \operatorname{height}(J).$$

By Theorem 1.4.18,

$$\mu(I) = \mu(\mathrm{H}_1(K^R)) = \mu(J).$$

Therefore, applying Corollary 0.3.10,

$$\begin{aligned} I \text{ is generated by a regular sequence } &\iff \mu(I) = \operatorname{height}(I) \\ &\iff \mu(J) = \operatorname{height}(J) \\ &\iff J \text{ is generated by a regular sequence.} \quad \square \end{aligned}$$

This settles the question of well-definedness of our definition of complete intersection.

Our next goal is to give another characterization of complete intersections. To do that, we will need some auxiliary tools.

The following idea is very helpful, and we might use variations of it on various occasions:

Remark 1.4.21. Let $(Q, \mathfrak{m}, k)$ be a regular local ring and let $R = Q/I$ for some ideal $I \subseteq \mathfrak{m}^2$. Fix minimal generators $\underline{y}$ for the maximal ideal of Q. Note that since Q is regular, the Koszul complex $\operatorname{Kos}^Q(\underline{y})$ is a minimal free resolution for k over Q, by Theorem 0.3.12. Let $Q[X]$ be a minimal model for R over Q. Since both $Q[X]$ and $\operatorname{Kos}^Q(\underline{y})$ are bounded below complexes of free modules, by Exercise 1.3.24 we get quasiisomorphisms of dg Q-algebras

$$K^R \xleftarrow{\simeq} \operatorname{Kos}(\underline{y}; Q) \otimes_Q R \xleftarrow{\simeq} \operatorname{Kos}^Q(\underline{y}) \otimes_Q Q[X] \xrightarrow{\simeq} k \otimes_Q Q[X] =: k[X].$$

Once we take homology, these all become isomorphisms, and in particular

$$\mathrm{H}(K^R) \cong \mathrm{H}(k[X]).$$

If we do not assume that R is complete, combining this argument with the quasiisomorphism $K^R \longrightarrow K^{\widehat{R}}$ from Theorem 1.4.14 gives us $\mathrm{H}(K^R) \cong \mathrm{H}(K^{\widehat{R}}) \cong \mathrm{H}(k[X])$.

On the other hand, the minimality condition on the minimal model (see Theorem 1.3.18) gives us an inclusion of kX_1 into $\mathrm{H}(k[X])$, which induces a map

$$\wedge_k(kX_1) = k[X_1] \longrightarrow \mathrm{H}(k[X]).$$

Moreover, note that $\wedge_k(\Sigma\,\mathrm{H}_1(K^R)) = k[e_1, \dots, e_n]$, where $e_1, \dots, e_n$ is a basis for $\mathrm{H}_1(K^R)$. The universal property of semifree extensions in Corollary 1.3.22 gives us a natural map

$$\wedge_k(\Sigma\,\mathrm{H}_1(K^R)) \longrightarrow \mathrm{H}(K^R).$$

Therefore, the zigzag above gives us the commutative diagram

$$\begin{tikzcd} \mathrm{H}(K^R) \arrow[r, "\cong"] & \mathrm{H}(k[X_1]) \\ \wedge_k(\Sigma\,\mathrm{H}_1(K^R)) \arrow[u] \arrow[r, "\cong"'] & k[X_1]. \arrow[u] \end{tikzcd}$$

We saw in Remark 1.2.8 that given a dg algebra, its homology inherits an algebra structure. In particular, we can consider the algebra $\mathrm{H}(K^R)$. The following characterization of complete intersections is part of the PhD thesis of E. F. Assmus JR, advised by John Tate.

Theorem 1.4.22 (Assmus)**.** *Let R be a noetherian local ring. The following are equivalent:*

1) *The ring R is a a complete intersection.*
2) *The natural map $\mathrm{H}(K^R) \longrightarrow \wedge_k(\Sigma\,\mathrm{H}_1(K^R))$ is an isomorphism of graded k-algebras.*
3) *There is a quasiisomorphism $K^R \simeq \wedge_k(\Sigma\,\mathrm{H}_1(K^R))$ of local dg algebras.*
4) *The k-vector space $\mathrm{H}_2(K^R)$ is generated by $\mathrm{H}_1(K^R)$, meaning that $\mathrm{H}_2(K^R) = \mathrm{H}_1(K^R)^2$.*

Proof. By Theorem 1.4.14, we can reduce to the complete case. Throughout, fix a regular presentation $(Q, \mathfrak{m}, k)$ for R, that is, assume $R = Q/I$ with $I \subseteq \mathfrak{m}^2$, and fix minimal generators $\underline{y}$ for the maximal ideal of Q. Let $Q[X]$ be a minimal model for R over Q, and let $\partial(X_1) = \{f_1, \dots, f_n\}$.

Suppose that R is a complete intersection. Then $\underline{f} = f_1, \dots, f_n$ is a regular sequence, and by Lemma 1.3.35, $Q[X] = Q[X_1] = \mathrm{Kos}^Q(\underline{f})$. Identifying $\underline{f}$ with its image in R and in k,

$$K^R \cong \mathrm{Kos}^Q(\underline{f}) \cong R \otimes_Q \mathrm{Kos}^Q(\underline{f})$$

and

$$k[X_1] = k[X] = k \otimes_Q \mathrm{Kos}^Q(\underline{f}) \cong \mathrm{Kos}^k(\underline{f})$$

has trivial differential. Therefore, $\mathrm{H}(k[X]) = k[X_1]$. Therefore, in the diagram we discussed in Remark 1.4.21, we have

$$\begin{tikzcd} \mathrm{H}(K^R) \arrow[r, "\cong"] & \mathrm{H}(k[X]) \\ \wedge_k(\Sigma\,\mathrm{H}_1(K^R)) \arrow[u] \arrow[r, "="] & k[X_1] \arrow[u, "\cong"'] \end{tikzcd}$$

We conclude that $\mathrm{H}(K^R) \cong \wedge(\mathrm{H}_1(K^R))$ and the map on the left is an isomorphism.

Note that 2) $\implies$ 3) is immediate. Moreover, 3) $\implies$ 4) follows from applying homology and using that that the exterior algebra $\wedge(\mathrm{H}_1(K^R))$ is generated in degree 1.

Now suppose that $\mathrm{H}_2(K^R) = \mathrm{H}_1(K^R)^2$. By Remark 1.4.21,

$$\mathrm{H}(K^R) \cong \mathrm{H}(k[X]).$$

Therefore, $\mathrm{H}_2(k[X]) = \mathrm{H}_1(k[X])^2$. By Exercise 1.3.14, in $Q[X]$ we have $\partial(X_2) \subseteq \mathfrak{m}X_1$, and thus all variables in X_2 become cycles in $k[X]$. Similarly, all variables in X_1 are cycles in $k[X]$. We conclude that $\mathrm{H}_1(k[X]) = k \cdot X_1$.

By Theorem 1.3.18, none of the variables in X_2 is a boundary in $k[X]$. In particular, any $x \in X_2$ contributes a nontrivial class in $\mathrm{H}_2(k[X])$. On the other hand, $(\mathrm{H}_1(k[X]))^2 = k \cdot X_1^2$, and $k \cdot X_1^2$ contains no variables of degree 2. We conclude that $X_2 = \emptyset$, and thus the minimal model $Q[X]$ for R over Q has no variables of degree 2. By Lemma 1.3.35, R must be a complete intersection. □

Exercise 1.4.23. Let $R = k[\![x,y]\!]/(x^2, xy)$. Find an explicit element in $\mathrm{H}_2(K^R)$ that is not in $(\mathrm{H}_1(K^R))^2$.

Problem 1.4.24 (Localization Problem for complete intersections)**.** Let R be a noetherian local ring. If R is a complete intersection, is R_P a complete intersection for all primes P?

Exercise 1.4.25. Let R be a *complete* noetherian local ring. Show that if R is a complete intersection, must R_P is a complete intersection for all primes P.

However, when R is not complete, the problem is quite delicate. The main issue is that completion and localization do not commute. Just like with the Localization Problem for regular local rings, we can solve this issue by giving an appropriate homological characterization of complete intersections. This problem was first solved by Avramov in 1977 [Avr77]. We will see an alternative proof later in these lectures.

1.5 Deviations

While the Betti numbers count the number of generators in each homological degree, there is a dg algebra analogue that counts the number of algebra generators we add in each degree. These are especially important for the acyclic closure of the residue field: as we will how in Theorem 1.5.17, the acyclic closure of is in fact the minimal free resolution for of the residue field.

Definition 1.5.1. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $R\langle Y\rangle$ be an acyclic closure of k. The **deviations** of R count the number of variables in each degree:

$$\varepsilon_i(R) := |Y_i|.$$

Remark 1.5.2. Let us compute the first few deviations of a noetherian local ring $(R, \mathfrak{m}, k)$. By definition,

$$\varepsilon_1(R) = \mu(\mathfrak{m}) = \operatorname{embdim}(R).$$

By construction, Y_2 maps bijectively onto $\mathrm{H}_1(R\langle Y_1\rangle) \cong \mathrm{H}_1(K^R)$, so by Theorem 1.4.18

$$\varepsilon_2(R) = \mu(I).$$

Remark 1.5.3. Let $R\langle Y\rangle$ be an acyclic closure k over R. By Lemma 1.4.15, the completion map induces quasiisomorphisms

$$R\langle Y_{\leqslant n}\rangle \longrightarrow R\langle Y_{\leqslant n}\rangle \otimes_R \widehat{R}.$$

Since $R\langle Y_{\leqslant n}\rangle$ are complexes of finitely generated R-modules, by Exercise 1.3.23 we can inductively show that we have quasiisomorphisms

$$R\langle Y_{\leqslant n}\rangle \otimes_R \widehat{R} \cong \widehat{R}\langle Y_{\leqslant n}\rangle.$$

Therefore, we get isomorphisms

$$\mathrm{H}_n(R\langle Y_{\leqslant n}\rangle) \cong \mathrm{H}_n(\widehat{R}\langle Y_{\leqslant n}\rangle).$$

In particular,

$$\varepsilon_n(R) = \varepsilon_n(\widehat{R}).$$

Remark 1.5.4. Given the deviations of a local ring R, we can compute the Betti numbers of k. Let us illustrate this by computing the first few. We know that $\beta_0(k) = 1$. As we will see in Theorem 1.5.17, the acyclic closure $R\langle Y\rangle$ is a minimal resolution for k, and $R\langle Y\rangle$ has the form

$$\cdots \longrightarrow \begin{array}{c} RY_4 \\ \oplus \\ RY_1Y_3 \\ \oplus \\ RY_1^2Y_2 \\ \oplus \\ RY_1^4 \\ \oplus \\ RY_2^{(2)} \end{array} \longrightarrow \begin{array}{c} RY_3 \\ \oplus \\ RY_1Y_2 \\ \oplus \\ RY_1^3 \end{array} \longrightarrow \begin{array}{c} RY_2 \\ \oplus \\ RY_1^2 \end{array} \longrightarrow RY_1 \longrightarrow R.$$

Thus

$$\beta_1(k) = \varepsilon_1(R) \qquad \beta_2(k) = \varepsilon_2(R) + \binom{\varepsilon_1(R)}{2} \qquad \beta_3(k) = \varepsilon_3(R) + \varepsilon_2(R)\varepsilon_1(R) + \binom{\varepsilon_1(R)}{3}.$$

This strategy allows us to find formulas for $\beta_i(k)$ in terms of $\varepsilon_j(R)$ for all i. Note that in these formulas, all the coefficients on $\varepsilon_j(R)$ are positive.

The deviations of R are closely related with the Poincaré series of k, the generating function for the Betti sequence of k.

Definition 1.5.5. Let R be a noetherian local ring and M a finitely generated R-module. The Poincaré series of M is the power series with integer coefficients given by

$$P_M^R(t) := \sum_{d=0}^{\infty} \beta_d(M)\, t^d.$$

Remark 1.5.6. When $M = R/I$ is a cyclic R-module, then $\beta_0(R/I) = 1$, and the Poincaré series of R/I has the form

$$1 + \sum_{i=1}^{\infty} b_i t^i.$$

Any power series of this form can be written uniquely as a (possibly infinite) product of the form

$$1+\sum_{i=1}^{\infty} b_i t^i = \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{e_{2i-1}}}{\prod_{i=1}^{\infty}(1-t^{2i})^{e_{2i}}}$$

that converges in the (t)-adic topology of $\mathbb{Z}[\![t]\!]$. This can be shown via a quick induction, going modulo (t^n) for each successive n to find e_n, which we leave as an exercise.

We claim that when we write the Poincaré series of the residue field k in this form, say

$$P_k^R(t) = \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{e_{2i-1}}}{\prod_{i=1}^{\infty}(1-t^{2i})^{e_{2i}}},$$

these exponents e_n are precisely the deviations $\varepsilon_n(R)$ of R.

To see this, let $R\langle Y\rangle$ be an acyclic closure for k. By Theorem 1.5.17, this is a minimal free resolution for k, so the differential in

$$k\langle Y\rangle := R\langle Y\rangle \otimes_R k$$

vanishes. Note that

$$k\langle Y\rangle = \bigotimes_{y\in Y} k\langle y\rangle.$$

Fix a particular variable $y \in Y$. If y has odd degree $2i-1$, then $k\langle y\rangle$ has a copy of k in degree 0 and another in degree $2i-1$, and nothing else, so

$$\sum_{n=0}^{\infty} \dim_k(k\langle y\rangle_n)\cdot t^n = 1+t^{2i-1}.$$

If y has even degree $2i$, then $k\langle y\rangle = k\langle y^{(i)} \mid i \geqslant 1\rangle$ has one copy of k in every degree that is a multiple of $2i$, and

$$\sum_{n=0}^{\infty} \dim_k(k\langle y\rangle_n)\cdot t^n = \sum_{\ell=0}^{\infty} t^{(2i)\ell} = \frac{1}{1-t^{2i}}.$$

To count the rank of $k\langle Y\rangle$ in degree n, we need only to count the number of monomials in the variables of Y of total degree n. Thus

$$P_k^R(t) = \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{|Y_{2i-1}|}}{\prod_{i=1}^{\infty}(1-t^{2i})^{|Y_{2i}|}} = \frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{\varepsilon_{2i-1}(R)}}{\prod_{i=1}^{\infty}(1-t^{2i})^{\varepsilon_{2i}(R)}}.$$

Lemma 1.5.7. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent:*

1) *R is regular.*

2) *$\varepsilon_n(R) = 0$ for all $n \geqslant 2$.*

3) *$\varepsilon_2(R) = 0$.*

Proof. If R is regular, by Theorem 0.3.12 the maximal ideal $\mathfrak{m}$ is generated by a regular sequence, and an acyclic closure of k is just the Koszul complex on a minimal generating set of $\mathfrak{m}$. Therefore, $\varepsilon_n(R) = 0$ for all $n \geqslant 2$.

Note that 2) $\Rightarrow$ 3) is obvious.

If $\varepsilon_2(R) = 0$, then the Koszul complex $R\langle Y_1\rangle$ on a minimal generating set for $\mathfrak{m}$ has $\mathrm{H}_1(R\langle Y_1\rangle) = 0$, so $\mathfrak{m}$ is generated by a regular sequence by Theorem 1.1.4. Therefore, R is regular by Theorem 0.3.12.

Alternatively, we can see that $I = 0$ by Remark 1.5.2, so $\widehat{R} \cong Q$ is a regular ring. This completes the proof since R is regular if and only if $\widehat{R}$ is regular. $\square$

Construction 1.5.8 (Avramov)**.** We saw in Remark 1.4.21 that there is a quasiisomorphisms of dg algebras

$$K^Q \otimes_Q Q[X] \xrightarrow{\simeq} k[X].$$

Let us focus on the right hand side and construct an acyclic closure of k over $k[X]$.

By Remark 1.4.21, we have

$$\mathrm{H}_0(k[X]) = k \qquad \text{and} \qquad \mathrm{H}_1(k[X]) = kX_1.$$

Set $Y_2 = \{y_x \mid x \in X_1\}$ to be a set of degree 2 variables, and consider $k[X]\langle y_x \mid \partial(y_x) = x\rangle$. By Exercise 1.3.29,

$$k[X]\langle \partial(y_x) = x\rangle \xrightarrow{\simeq} k[X]/(X_1) \cong k[X_{\geqslant 2}].$$

Hence

$$\mathrm{H}_n(k[X]\langle \partial(y_x) = x\rangle) = \begin{cases} k & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ kX_2 & \text{if } n = 2. \end{cases}$$

Now we repeat this idea in degree 3: by Exercise 1.3.23, there is a quasiisomorphism

$$k[X]\langle Y_2, Y_3\rangle \xrightarrow{\simeq} k[X_{\geqslant 2}]\langle Y_3 \mid \partial(y_x) = x\rangle.$$

Fix $x \in X_2$. Lifting $\partial(y_x)$ under this quasiisomorphism, we see that in $k[X]\langle Y_2, Y_3\rangle$

$$\partial(y_x) = x + a + y$$

for some $a \in kX_1^2$ and $y \in kY_2$. We claim that $y = 0$. Indeed, applying ∂ again, we get

$$0 = \partial^2(y_x) = \partial(x) + \partial(a) + \partial(y).$$

Note that $\partial(x) = 0$, since $\partial(X_2) = 0$ in $k[X]$ by Exercise 1.3.14. Moreover, $\partial(a) = 0$ since by Exercise 1.3.14 and the Leibniz rule $\partial(X_1^2) = 0$ in $k[X]$. Therefore, $\partial(y) = 0$. However, note that kX_2 maps bijectively onto kX_1 under the differential:

$$\partial\colon kY_2 \xrightarrow{\simeq} kX_1.$$

Thus $y = 0$. We now conclude that

$$\partial(y_x) = x + a_x \qquad \text{for some } a_x \in (X)^2.$$

We now continue inductively, and construct an acyclic closure over $k[X]$, which has the form

$$k[X]\langle Y_{\geqslant 2}\rangle \xrightarrow{\simeq} k$$

with

$$Y_n = \{y_x \mid x \in X_{n-1}\} \qquad \text{and} \qquad |y_x| = n \text{ for each } n \geqslant 2$$

and

$$\partial(y_x) = x + a_x \qquad \text{with} \qquad a_x \in (X)(X, Y^{(n)} \mid n \geqslant 1).$$

Theorem 1.5.9. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Fix an acyclic closure $R\langle Y\rangle$ for k and a minimal model $Q[X]$ for R. There are quasiisomorphisms of local dg algebras*

$$R\langle Y_{\leqslant n}\rangle \xrightarrow{\simeq} k[X_{\geqslant n}] =: k[X]/(X_{<n}).$$

Proof. We saw in Remark 1.4.21 that there is a zigzag of quasiisomorphisms of dg algebras

$$K^R \xleftarrow{\simeq} K^Q \otimes_Q Q[X] \xrightarrow{\simeq} k[X].$$

On the right hand side, the quaiisomorphism we built in Construction 1.5.8 together with Exercise 1.3.23 give us a quasiisomorphism

$$K^Q \otimes_Q Q[X]\langle Y_{\geqslant 2}\rangle \xrightarrow{\simeq} k[X]\langle Y_{\geqslant 2}\rangle \xrightarrow{\simeq} k.$$

Moreover, using Exercise 1.3.23 on the right hand side of our zigzag, we also get a quasiisomorphism, leading to a new zigzag of quasiisomorphisms

$$K^R\langle Y_{\geqslant 2}\rangle \xleftarrow{\quad\simeq\quad} K^Q \otimes_Q Q[X]\langle Y_{\geqslant 2}\rangle \xrightarrow{\quad\simeq\quad} k[X]\langle Y_{\geqslant 2}\rangle \xrightarrow{\quad\simeq\quad} k. \qquad \square$$

Remark 1.5.10. Let R be a noetherian local ring, and $\widehat{R} \cong Q/I$ with $(Q, \mathfrak{m}, k)$ a minimal regular presentation. Let $Q[X]$ be a minimal model for R over Q, set $k[X] := Q[X] \otimes_Q k$, and consider the quotient complexes $k[X_{\geqslant n}]$ attained by modding out by the dg ideal $(X_{\leqslant n-1})$. Since $Q[X]$ is a minimal model, in $k[X_{\geqslant n}]$ we have $\partial(X_{n+1}) = 0$ and $\partial(X_n) = 0$. Moreover,

$$k[X_{\geqslant n}]_n = kX_n.$$

Therefore,

$$\mathrm{H}_n(k[X_{\geqslant n}]) = kX_n.$$

Theorem 1.5.11 (Avramov, 1984 [Avr84]). *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Fix an acyclic closure $R\langle Y\rangle$ for k and a minimal model $Q[X]$ for R. Then for all $i \geqslant 2$,*

$$\varepsilon_i(R) = |Y_i| = |X_{i-1}|.$$

Proof. By Theorem 1.5.9, we have quasiisomorphisms

$$R\langle Y_{\leqslant n}\rangle \xrightarrow{\simeq} k[X_{\geqslant n}] =: k[X]/(X_{<n}).$$

By Remark 1.5.10

$$\mathrm{H}_n(k[X_{\geqslant n}]) = kX_n.$$

Therefore,

$$\mathrm{H}_n(R\langle Y_{\leqslant n}\rangle) \cong \mathrm{H}_n(k[X_{\geqslant n}]) = kX_n$$

is minimally generated by $|X_n|$ many elements. Since the variables in Y_{n+1} are added to kill degree n cycles in $R\langle Y_{\leqslant n}\rangle$, we conclude that $|Y_{n+1}| = |X_n|$. □

We can now think about deviations in two ways: via the acyclic closure of k or via the minimal model of R.

Remark 1.5.12. We saw in Remark 1.5.2 that $\varepsilon_1(R) = \operatorname{embdim}(R)$ and $\varepsilon_2(R) = \mu(I)$. We can now also compute $\varepsilon_3(R)$.

Since $Q[X_1]$ is the Koszul complex on a minimal generating set $\underline{f}$ for I, the number of variables in X_2 is the minimal number of generators for the first Koszul homology on $\underline{f}$. Since the Koszul homology is independent of the choice of minimal generators for I, we simply write this as $\mathrm{H}_1(\mathrm{Kos}(I))$. Thus

$$\varepsilon_3(R) = \mu(\mathrm{H}_1(\mathrm{Kos}(I))).$$

Theorem 1.5.13. *A noetherian local ring R is a complete intersection if and only if*

$$\varepsilon_n(R) \;=\; 0 \qquad \textit{for all} \qquad n \geqslant 3.$$

Proof. Let $\widehat{R} \cong Q/I$ with $(Q, \mathfrak{m}, k)$ a minimal regular presentation for R, and let $Q[X]$ be a minimal model for R over Q.

If I is generated by a regular sequence, then $Q[X] = Q[X_1]$ by Lemma 1.3.35. By Theorem 1.5.11, we can read the deviations of R from the minimal model. Therefore, for all $n \geqslant 3$ we have

$$\varepsilon_n(R) = |X_{n-1}| = 0. \qquad \square$$

Conversely, if $\varepsilon_n(R) = 0$ for all $n \geqslant 3$, then by Theorem 1.5.11 we have $X_n = \emptyset$. By Lemma 1.3.35, R must be a complete intersection.

Tate [Tat57] gave a very different proof of Theorem 1.5.13 by showing directly that the acyclic closure $R\langle Y\rangle$ of k over R has $Y_n = 0$ for all $n \geqslant 3$.

Exercise 1.5.14. Let $R = Q/I$ with $(Q, \mathfrak{m}, k)$ a regular local ring and $I \subseteq \mathfrak{m}^2$ generated by a regular sequence. Use the quasiisomorphisms of local dg algebras (with residue field k)

$$K^R \xleftarrow{\simeq} Q[X_1] \otimes_Q K^Q \xrightarrow{\simeq} k[X_1].$$

and Exercise 1.3.29 to write down an explicit acyclic closure of k over R in terms of the following data:[2] $\mathfrak{m}$ is generated by $t_1, \dots, t_n$, the ideal I is minimally generated the regular sequence $f_1, \dots, f_c$, and

$$f_i = \sum_{j=1}^{n} a_{ij} t_j \quad \text{for some} \quad a_{ij} \in \mathfrak{m}\,.$$

In fact, the vanishing of the higher deviations characterizes complete intersections. The characterization of complete intersections we will state next puts together the work of several authors. The cumulative theorem tells us that various conditions are equivalent to being a complete intersection: Assmus [Ass59] showed the equivalence with (3) and Gulliksen showed the equivalence with conditions (4) [Gul71] and (5) [Gul80]. The last condition, due to Halperin [Hal87] and known as Halperin's Rigidity Theorem, is the most amazing: as long as at least one deviation vanishes, then R must be a complete intersection. This tells us that as long as R is not a complete intersection, then when constructing a minimal model for R over Q or an acyclic closure for k over R, we must add new variables in *every* degree.

Theorem 1.5.15 (Assmus, 1959 [Ass59], Gulliksen, 1971 [Gul71] and 1980 [Gul80], Halperin, 1987 [Hal87])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent:*

1) R is a complete intersection.

2) $\varepsilon_n(R) = 0$ for all $n \geqslant 3$.

3) $\varepsilon_3(R) = 0$.

4) $\varepsilon_n(R) = 0$ for all $n \gg 0$.

5) $\varepsilon_{2n}(R) = 0$ for all $n \gg 0$.

6) $\varepsilon_n(R) = 0$ for some $n \geqslant 1$.

Remark 1.5.16. Let $Q[X]$ be a minimal model for R over Q and $R\langle Y\rangle$ an acyclic closure for k over R. By Theorem 1.5.15 and Theorem 1.5.11, if R is not a complete intersection, then $X_i \neq \emptyset$ and $Y_i \neq \emptyset$ for all $i \geqslant 1$.

Theorem 1.5.17 (Gulliksen, 1968 [Gul68], Schoeller, 1967 [Sch67])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. An acyclic closure for k is a minimal free resolution for k.*

[2] This description is due to Tate.

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Nomenclature

$[x]$	homology class of the cycle x
$\mathrm{depth}(M)$	depth of M (wrt the maximal ideal)
$\mathrm{embdim}(R)$	embedding dimension of R, given by $\mu(\mathfrak{m})$
$\mathrm{Kos}(x_1, \dots, x_n)$	the Koszul complex on $x_1, \dots, x_n$
$\mu(M)$	minimal number of generators of M, given by $\dim_k(M/\mathfrak{m}M)$
$\mathrm{codepth}(R)$	codepth of R, defined as $\mathrm{embdim}(R) - \mathrm{depth}(R)$
$\mathrm{pdim}_R(M)$	projective dimension of M over R
$\sim$	We write $f \sim g$ to indicate f and g are homotopic maps
ΣC	shift of C, with $(\Sigma C)_n = C_{n-1}$
$\mathrm{Syz}_n(M)$	nth syzygy of M
$\underline{x}$	shorthand for $x_1, \dots, x_n$
K^R	Koszul complex on a minimal generating set for $\mathfrak{m}_R$
RLR	regular local ring

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