Homological Methods in Commutative Algebra

Problem Set 1

Throughout, all rings all commutative and noetherian, and all modules are finitely generated.

Problem 1. Write the Koszul complex on 3 elements f_1, f_2, f_3.

Problem 2. Let I be a proper nonzero ideal in noetherian local ring R. Show that

$$\beta_i(I) = \beta_{i+1}(R/I)$$

for all $i \geqslant 0$. In particular, note that I has finite projective dimension if and only if R/I has finite projective dimension.

Problem 3. Check that if M is a free module over a domain, then $\operatorname{rank}(M)$ is the free rank of M.

Problem 4. Let R be a local/graded domain and M be a finitely generated (graded) R-module.

a) Show that if F is any finite free resolution for M over R, then

$$\sum_i (-1)^i \operatorname{rank}(F_i) = \operatorname{rank}(M).$$

In particular,

$$\sum_i (-1)^i \beta_i(M) = \operatorname{rank}(M).$$

b) Show that if I is a nonzero proper ideal of R and M is an R/I-module with $\operatorname{pdim}_R(M) < \infty$, then for any finite free resolution F for M over R

$$\sum_{i \geqslant 0} \operatorname{rank}(F_{2i}) = \sum_{i \geqslant 0} \operatorname{rank}(F_{2i+1}).$$

Problem 5. Show that for all finitely generated modules M over a domain R,

$$\beta_i(M) = \operatorname{rank}\left(\Omega_i(M)\right) + \operatorname{rank}\left(\Omega_{i+1}(M)\right).$$

Problem 6. Let $Q = k[\![x, y, z, w]\!]$, $I = (xy, yz, zw)$, and $M = Q/I$.

a) Find $\operatorname{pdim}(M)$ without writing the minimal free resolution for M.

b) Find the betti numbers of M without writing the minimal free resolution for M.

c) Find the minimal free resolution for M.

d) Check your work with Macaulay2.

Problem 7. Let M be a finitely generated R-module. Assume that either $(R, \mathfrak{m}, k)$ is a noetherian local ring or that R is a standard graded finitely generated algebra over a field $k = R_0$, in which case M is graded.

a) Show that

$$\beta_i(M) = \dim_k\left(\operatorname{Tor}_i^R(M, k)\right) = \dim_k\left(\operatorname{Ext}_R^i(M, k)\right).$$

b) In the graded case, show that

$$\beta_{i,j}(M) = \dim_k\left(\operatorname{Tor}_i^R(M, k)_j\right) = \dim_k\left(\operatorname{Ext}_R^i(M, k)_{-j}\right).$$

Problem 8. Let R be a noetherian local ring and let M and N be finitely generated R-modules. Show that for all $i \geqslant 1$,

$$\operatorname{Tor}_{i+1}^R(M, N) \cong \operatorname{Tor}_i^R(\Omega_1 M, N).$$

Problem 9 (Localization Problem)**.** Let R be a regular local ring. Show that for all prime ideals P, the localization R_P is a regular local ring.

Problem 10. Show that $\beta_2(R/I)$ can be arbitrarily large for 3-generated ideals in $R = k[\![x_1, \ldots, x_d]\!]$ with $d \geqslant 3$: for all $N \geqslant 1$ there exists a 3-generated ideal I such that $\beta_2(R/I) \geqslant N$.

Problem 11. Let $M \neq 0$ be a finitely generated module over a noetherian local ring containing a field, and let $p = \operatorname{pdim}(M) < \infty$. Show that

$$\beta_i(M) \geqslant \begin{cases} 2i+1 & \text{if } i < p-1 \\ p & \text{if } i = p-1 \\ 1 & \text{if } i = p. \end{cases}$$

Problem 12. Let $I \neq R$ be a radical ideal in a regular ring R, and set

$$c := \max\{\operatorname{height} P \mid P \in \operatorname{Min}(I)\}.$$

Show that for all i,

$$\beta_i(R/I) \geqslant \binom{c}{i}.$$

Problem 13. Let k be a field and consider an exact sequence of k-vector spaces $A \longrightarrow B \longrightarrow C$. Show that

$$\dim_k(B) \leqslant \dim_k(A) + \dim_k(C).$$

Problem 14. Let $(R, \mathfrak{m})$ be a noetherian local ring and $M = R/I$ be a cyclic module of projective dimension 1. Show that $I = (f)$ and $f \in \mathfrak{m}$ is a regular element on R.

Problem 15. Let $R = k[\![x_1, \ldots, x_d]\!]$ with k a field and let I be a proper ideal in R.

a) When $\mu(I) = 2$, what are the possible values for $\operatorname{depth}(R/I)$?

b) When $\mu(I) = 3$, what are the possible values for $\operatorname{depth}(R/I)$?

Problem 16. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Show that $\operatorname{pdim}(M) \leqslant \operatorname{pdim}(k)$ for all finitely generated R-modules M.

Problem 17. Let k be any field and $R = k[x, y, z_1, \ldots, z_n]$ for some $n \geqslant 3$. Show that the ideal

$$I = \left(x^n, y^n, \sum_{i=0}^{n-1} z_{i+1} x^i y^{n-i} \right)$$

has $\operatorname{pdim}(R/I) = n + 2$.

Problem 18. Let $(Q, \mathfrak{m}, k)$ be a regular local ring, and let $R = Q/I$ be Cohen-Macaulay with $\operatorname{pdim}_Q(R) = n$. Show that R is Gorenstein if and only if the betti sequence for R over Q is symmetric, meaning that for all i

$$\beta_i^Q(R) = \beta_{n-i}^Q(R).$$

Note: this problem requires using an idea we will not discuss in these lectures.

Let k be a field and let $f_1, \ldots, f_n$ be monomials in $k[x_1, \ldots, x_d]$, minimally generating the ideal $I = (f_1, \ldots, f_n)$. For each subset $J \subseteq [n] := \{1, \ldots, n\}$, set

$$f_J = \operatorname{lcm}(f_j \mid j \in J).$$

The **Taylor resolution** of R/I is the complex (T, ∂) defined as follows:

- In homological degree s, T_s is the free R-module on basis e_J with

$$J = \{j_1, \ldots, j_s\} \subseteq [n]$$

  ranging over all the subsets of $[n]$ of size $|J| = s$.

- For each basis element e_J with $|J| = s$, the differential is defined as

$$\partial(e_J) = \sum_{i=1}^{s} (-1)^{i+1} \frac{f_J}{f_{J \setminus \{j_i\}}} e_{J \setminus \{j_i\}}.$$

In the 1960s, in her PhD thesis, Diana Taylor proved that this is a free resolution for R/I, which is now known as the **Taylor resolution**. We will use her theorem without proof.

Note: any monomial ideal I has a unique minimal generating set consisting of monomials, sometimes denoted $G(I)$.

Problem 19. Let k be a field and consider the monomial ideal $I = (xy, xz, yz) \subseteq k[x, y, z]$.

a) Find the Taylor resolution for R/I.

b) Use the Taylor resolution to find the minimal free resolution for R/I.

Problem 20. Consider a field k, $Q = k[x, y, z, w]$, and

$$I = (x^2, xy, yz, zw, w^2).$$

a) Find the Taylor resolution for R/I.

b) Use the Taylor resolution to find the betti numbers for R/I.

c) Check your work with Macaulay2.

Problem 21. Let $I = (f_1, \ldots, f_n)$ be a squarefree monomial ideal. Show that the Taylor resolution for R/I is minimal if and only if for each i there exists a variable y_i such that $y_i \mid f_i$ but $y_i \nmid f_j$ for all $j \neq i$.