Homological Methods in Commutative Algebra

Problem Set 2

Throughout, all rings all commutative and noetherian, and all modules are finitely generated.

Problem 1. Let I be a nonzero proper ideal in a noetherian domain R and let $f_1, \dots, f_c$ be a maximal regular sequence inside I. Consider the short exact sequence

$$0 \longrightarrow N \longrightarrow R/(f_1, \dots, f_c) \xrightarrow{\pi} R/I \longrightarrow 0.$$

where π is the canonical quotient map.

a) Show that $\operatorname{Ext}_R^{c-1}(N, R) = 0$.

b) Show that the induced map

$$\pi^* = \operatorname{Ext}_R^c(\pi, R) \colon \operatorname{Ext}_R^c(R/I, R) \longrightarrow \operatorname{Ext}_R^c(R/(f_1, \dots, f_c), R)$$

is nonzero.

Problem 2. Let $Q = k[\![x, y]\!]$, $I = (x^2, xy)$, and $R = Q/I$.

a) Write the first 3 steps to construct a minimal model for R over Q.

b) Write the first 3 steps to construct an acyclic closure for k over R.

c) Check your work with Macaulay2.[1]

Problem 3. Let $(R, \mathfrak{m}, k)$ be any noetherian local ring of dimension d. Show that

$$\beta_i(k) \geqslant \binom{d}{i}.$$

Let k be a field and let $f_1, \dots, f_n$ be monomials in $k[x_1, \dots, x_d]$, minimally generating the ideal $I = (f_1, \dots, f_n)$. The Taylor resolution of R/I has a dg algebra structure, defined on basis elements as follows:

$$e_J \cdot e_L = \operatorname{sgn}(J, L) \, \frac{f_J f_L}{f_{J \cup L}} \, b_{J \cup L} \, .$$

Here the sign is given by

$$\operatorname{sgn}(J, L) = (-1)^{\varepsilon} \quad \text{where} \quad \varepsilon = |\{(j, \ell) \, : \, j \in J, \ \ell \in L, \text{ and } j > \ell\}|$$

and by convention, if $J \cap L \neq \varnothing$, then $\operatorname{sgn}(J, L) = 0$.

Problem 4. Let $I = (f_1, \dots, f_n)$ be a monomial ideal in $Q = k[x_1, \dots, x_d]$ and consider its Taylor resolution T. Fix a subset $J \subseteq [n]$.

a) Show that

$$\partial(e_J) \notin \mathfrak{m}T \iff f_i \mid f_{J \setminus \{i\}} \text{ for some } i \in J.$$

b) Show that for each $i \in [n]$,

$$e_i e_J \notin \mathfrak{m}T \iff \gcd(f_i, f_j) = 1 \text{ for all } j \in J.$$

[1] Try out the `DGAlgebras` package by Frank Moore and Keller VandeBogert and the `acyclicClosure` function.

Problem 5. Let k be any field and $R = k[x, y, z_1, \ldots, z_n]$ for some $n \geqslant 3$. Show that the ideal

$$I = \left(x^n, y^n, \sum_{i=0}^{n-1} z_{i+1} x^i y^{n-i} \right)$$

has $\operatorname{pdim}(R/I) = n + 2$.