Homological Methods in Commutative Algebra

Problem Set 3

Throughout, all rings all commutative and noetherian, and all modules are finitely generated.

Problem 1. Let $Q = k[\![x, y]\!]$, $I = (x^2, xy)$, and $R = Q/I$.

a) Write the first 3 steps to construct a minimal model for R over Q.
b) Write the first 3 steps to construct an acyclic closure for k over R.
c) Check your work with Macaulay2 over your favorite field of characteristic 0.[1]

Problem 2. Let $(R, \mathfrak{m}, k)$ be any noetherian local ring of embedding dimension d. Show that

$$\beta_i(k) \geqslant \binom{d}{i}.$$

Problem 3. Show that if R is a complete intersection of codimension c, then every finitely generated R-module has complexity at most c.

Problem 4. Let k be a field, $Q = k[\![x, y]\!]$, and $I = (x^2, xy)$. Find a system of higher homotopies for xy on the minimal free resolution for Q/I.

Problem 5. Let k be a field, $Q = k[\![x, y]\!]$, and $R = Q/(x^2, xy)$. Find a system of higher homotopies for x^2, xy on the minimal free resolution for R over Q.

Problem 6. Let k be a field, $Q = k[\![x, y]\!]$, and $R = Q/(x^2)$. Use the Shamash construction to find a resolution for $M = R/(xy)$ over R.

Problem 7. Let Q be a regular local ring and let $R = Q/I$ with I minimally generated by $\underline{f} = f_1, \ldots, f_n$. Let F be a free resolution of R over Q that has a structure of a DG algebra. Let $e_1, \ldots, e_n$ be a basis for F_1 with $\partial(e_i) = f_i$. Show that we get a system of higher homotopies $\{\sigma_\omega\}$ for $\underline{f}$ on F by setting

$$\sigma_{\mathbf{e}_i}(-) = e_i \cdot - \qquad \text{and} \qquad \sigma_\omega(u) = 0 \text{ for all } |\omega| \geqslant 2.$$

Problem 8. Let $(Q, \mathfrak{m}, k)$ be a regular local ring and $R = Q/(f)$ with $f \in \mathfrak{m}^2$. Show that under an appropriate choice choice of resolution and system of higher homotopies, the Shamash construction leads to the minimal free resolution for k over R.

[1] Try out the `DGAlgebras` package by Frank Moore and Keller VandeBogert and the `acyclicClosure` function.